CERN-PH-TH/2007-084

Topological amplitudes in heterotic strings with Wilson lines

Marlene Weiss[a]

Department of Physics, CERN
Geneva 23, CH-1211 Switzerland

and

ETH Zurich
CH-8093 Zürich, Switzerland

Abstract

We consider d=4, $\mathcal{N}=2$ compactifications of heterotic strings with an arbitrary number of Wilson lines. In particular, we focus on known chains of candidate heterotic/type II duals. We give closed expressions for the topological amplitudes $F^{(g)}$ in terms of automorphic forms of $SO(2+k,2,\mathbb{Z})$, and find agreement with the geometric data of the dual K3 fibrations wherever those are known.

[a] marlene.weiss@cern.ch

Contents

1 Introduction

Over the last decade, tremendous progress has been made in establishing and understanding $d = 4$, $\mathcal{N} = 2$ heterotic-type II duality, which connects the heterotic string compactified on $K3 \times \mathbb{T}^2$ with compactifications of type IIA theory on $K3$-fibrations. One of the most fruitful approaches has been to compute the low energy effective action for models with explicitly known heterotic and type II realizations. More precisely, the 4d effective action of these $\mathcal{N} = 2$ compactifications has been known for a long time to contain a series of BPS protected higher-loop terms of the form

$$S \sim \int F^{(g)}(t, \bar{t}) T^{2g-2} R^2 + \cdots , \tag{1.1}$$

where R is the Riemann tensor, T the graviphoton field strength, and the couplings $F^{(g)}$ are amplitudes of the topological string on the internal Calabi-Yau [1, 2]. On the heterotic side, these amplitudes appear at 1–loop [3] and are therefore in general accessible to computation [4, 5, 6, 7]. The result can be mapped to the type II side, yielding striking predictions in enumerative geometry.

The amplitudes $F^{(g)}$ are also intriguing from a mathematical point of view, as they involve interesting classes of automorphic functions. Furthermore, the Higgs transitions on the heterotic side correspond to geometric transitions between the corresponding Calabi-Yaus on the type II side. A more precise picture of how the heterotic moduli spaces are connected might therefore provide some insight into the web of type II vacua.

Until now, most explicit comparisons between heterotic and type II models have been restricted to cases with a small number n_v of massless Abelian vector multiplets, namely $n_v = 3, 4, 5$. These vector multiplets are the graviphoton, the heterotic dilaton S, one or two ($n_v = 4$) moduli T, U from the compactification torus, and if $n_v = 5$, one Wilson line modulus V. However, by now there is a myriad of conjectured heterotic-type II pairs with higher numbers of vector multiplets waiting to be analyzed.

In [8], the authors obtained chains of heterotic-type II duals by compactifying the heterotic string on $K3 \times \mathbb{T}^2$ in various orbifold realizations. In each chain, subsequent models are connected by a sequential Higgs mechanism reducing the number of generic Wilson line moduli by one. $K3$ is realized as an orbifold $\mathbb{T}^4/\mathbb{Z}_N$, $N = 2, 3, 4, 6$ and the $\mathbb{Z}_N$ is simultaneously embedded in the gauge connection in a modular invariant way. For the last models in the chains, the candidate type II duals can be explicitly constructed.

The classical vector multiplet moduli space of compactifications with $k = n_v - 4$ Wilson lines is given by the special Kähler space

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2+k,2)}{SO(2+k) \times SO(2)}, \tag{1.2}$$

where the first factor corresponds to the dilaton and the second to the torus and Wilson line moduli. The T-duality group, under which the vector multiplet couplings have to transform as automorphic functions, is $SO(2 + k, 2; \mathbb{Z})$ [9, 10, 11].

For the $SO(2, 2; \mathbb{Z})$ case with four vector multiplets, i.e. the well-known STU model, the higher derivative couplings have been computed in [5]. They can be expressed in terms of expansion coefficients of ordinary modular forms. The case with five vector multiplets (one Wilson line) has been studied at the level of prepotential and $F^{(1)}$ in [12]. This case is somewhat special, as the T-duality group is here $SO(3, 2; \mathbb{Z}) \cong Sp(4, \mathbb{Z})$ [11], and the corresponding automorphic functions are given by Siegel modular forms [11]. The effective couplings can be expressed in terms of Jacobi forms of index one, yielding a prescription how to split off the part depending on the Wilson line modulus from the gauge lattice.

The generic case involves more general automorphic forms. However, we can define a splitting procedure analogous to the one in [12], and the split lattice sum can be explicitly expressed in terms of ordinary Jacobi Theta functions. Once this split is determined, we can use the technique of lattice reduction [13] to explicitly compute higher-derivative F-terms for heterotic $\mathcal{N} = 2$ compactifications with an arbitrary number of Wilson lines. The final result involves the q-expansion coefficients of the moduli independent Higgsed part of the lattice sum. Even though the computation is done at the orbifold point, the results are fully valid at generic points of $K3$ moduli space, since the couplings $F^{(g)}$ only

depend on vector multiplets and therefore cannot mix with the $K3$ moduli, belonging to hypermultiplets.

While the formalism can be applied to almost any symmetric $\mathbb{Z}_N$ orbifold limit of $K3$, we mainly focus on the dual pairs found in [8]. We compute the corresponding topological amplitudes $F^{(g)}$ in closed form. For genus zero, our results agree with the numbers of rational curves found on the type II side wherever those are known [14]. The present computation extends previous work on threshold corrections for models with a single Wilson line [15, 12, 16], and also provides a more explicit realization, extended to higher genus, of the general results of [17].

This paper is organized as follows. In section 2, we review heterotic compactifications with $\mathcal{N} = 2$ supersymmetry and the Higgs chains of [8]. In section 3, we explain how to compute partition sums and higher derivative F-terms in general heterotic orbifold setups. Section 4 introduces the lattice splits in the presence of Wilson lines. A general expression for the amplitudes $F^{(g)}$ in the presence of Wilson lines is derived. In section 5, we use our results to extract geometric information on the dual Calabi-Yau manifold. This provides a highly nontrivial check of our computation in those cases where instanton numbers are known on the type II side. Section 6 contains some concluding remarks and further directions of research. Appendix A summarizes some facts about Jacobi and Riemann-Siegel theta functions, and appendix B reviews the Borcherds-Harvey-Moore technique of lattice reduction. Finally, appendix C collects tables of instanton numbers for several models discussed in the text.

2 Heterotic $\mathcal{N} = 2$ compactifications

In this section, we briefly discuss the construction of heterotic $\mathcal{N} = 2$ compactifications and their matter spectrum. There are two main approaches to analyzing these models. Section 2.1 reviews the purely geometrical approach of [18], while section 2.2 reviews the exact CFT construction via orbifolds of [8]. Even though the two approaches are completely equivalent, it proves very useful to keep the two in mind simultaneously, as sometimes one is more convenient, sometimes the other. Section 2.3 reviews how these compactifications fall into chains of models connected by a sequential Higgs mechanism [8].

2.1 The Calabi-Yau approach

Consider compactification of the heterotic string on $K3\times\mathbb{T}^2$. In order to break the gauge group $\mathcal{G} = E_8 \times E_8$ of the ten-dimensional heterotic string down to a subgroup G, one gives gauge fields on K3 an expectation value in H, where $G \times H$ is a maximal subgroup of $\mathcal{G}$. Geometrically, this corresponds to embedding a H-bundle V on K3. This bundle can be chosen to be the tangent bundle of K3, an $SU(2)$-bundle with instanton number $\int_{K3} c_2(V) = 24$. This is the standard embedding, where the spin connection on K3 is equal to the gauge connection. More generally, one can embed several stable holomorphic SU(N)-bundles V_a, as long as the constraints from modular invariance

$$\sum_a c_2(V_a) = 24 \qquad c_1(V_a) = 0 \tag{2.1}$$

are satisfied. We will here only consider embeddings of one or two $SU(2)$-bundles on one respectively both E_8 and write their instanton numbers according to (2.1) as $(d_1, d_2) = (12+n, 12-n)$.
The number of gauge neutral hypermultiplets is determined as follows [18]. There is a universal gravitational contribution of 20, and each of the $SU(N_a)$-bundles $V_a \to K3$ with $\int_{K3} c_2(V_a) = A$ has an extra $AN_a + 1 - N_a^2$ moduli, therefore we get additional 45 moduli for one and 51 for two embedded $SU(2)$ bundles. The rank of the gauge group is reduced by the rank of the embedded bundle, N-1. For the standard embedding, we thus find 65 hypermultiplets and an enhanced gauge group $E_7 \times E_8$, the first model in the $\mathbb{Z}_2$ chain in [8]. The Cartan subalgebra of $E_7 \times E_8$ contains 15 generators, and there is an extra $U(1)^4$ from the SUGRA multiplet and torus compactification, therefore this model has $n_v = 19$ vector multiplets.

2.2 Exact CFT construction via orbifolds

Rather than following the approach presented above, we will here realize the heterotic models following [8] in the so-called exact CFT construction via orbifolds. In this approach, the K3 is realized as a $\mathbb{Z}_N$ orbifold, while simultaneously the spin connection is embedded into the gauge degrees of freedom. We will mainly concentrate on the $\mathbb{Z}_N$–embeddings given in table 2.1. The orbifold $\mathbb{Z}_N$ twist θ acts on two of the four complex bosonic transverse coordinates as $\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{N}}$. Since we impose $\mathcal{N}=2$ SUSY, N can only take on the values $2, 3, 4, 6$ [17]. The action of θ on the gauge degrees of freedom is strongly restricted by worldsheet modular invariance. We implement it as a shift of the gauge lattice, writing for the torus and gauge lattice sum

$$\mathbf{Z}^{18,2}\left[{a \atop b}\right] = \sum_{p\in\Gamma^{18,2}+a\gamma} \mathrm{e}^{2\pi\mathrm{i}b\gamma\cdot p} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}}, \tag{2.2}$$

where $a, b \in \{1/N, \cdots (N-1)/N\}$. The shift $\gamma \in \Gamma^{18,2}$ has to fulfill the modular invariance and level-matching constraints [19]

$$\sum_{i=1}^{8} \gamma_i = \sum_{i=9}^{16} \gamma_i = 0 \bmod 2 \tag{2.3}$$

and

$$\gamma^2 = 2 \bmod 2N. \tag{2.4}$$

One then finds the possible inequivalent $\mathbb{Z}_N$ orbifolds: There are 2 for $\mathbb{Z}_2$, 5 for $\mathbb{Z}_3$, 12 for $\mathbb{Z}_4$ and 61 for $\mathbb{Z}_6$ [16]. Note that in those cases where the same type of shift is modular invariant for different N, those models are equivalent as far as the topological amplitudes $F^{(g)}$ are concerned. The reason for this is that they are only distinguished by the specific orbifold realization of the K3-surface. Since the moduli of the K3 live in hypermultiplets which do not mix with the vector multiplets, the higher-derivative couplings should be identical for the different $\mathbb{Z}_N$ embeddings. They can however differ if we turn on Wilson line moduli corresponding to the gauge groups only present in the orbifold limit [17], as will be explained in section 4.2.

Some non-standard embeddings, along with their perturbative gauge group, are given in table 2.2. These groups are easily read off from the simple root system for E_8 given below, table 2.2. The unbroken group is generated by the roots α_i invariant under the shift γ, i.e. fulfilling

$$\mathrm{e}^{\frac{2\pi \mathrm{i}\gamma\cdot\alpha_i}{N}} = 1. \tag{2.5}$$

In the first embedding in table 2.2, the invariant roots on the first E_8 are the 126 roots

$\mathbb{Z}_2$	γ^1=(1,-1,0,0,0,0,0,0); γ^2=(0,0,0,0,0,0,0,0)	$SU(2) \times E_7 \times E_8'$	n=12
$\mathbb{Z}_3$	γ^1=(1,1,2,0,0,0,0,0); γ^2=(1,-1,0,0,0,0,0,0)	$SU(3) \times E_6 \times U(1)' \times E_7'$	n=6
$\mathbb{Z}_4$	γ^1=(1,1,1,-3,0,0,0,0); γ^2=(1,1,-2,0,0,0,0,0)	$SO(10) \times SU(4) \times E_6' \times SU(2)' \times U(1)'$	n=4
$\mathbb{Z}_6$	γ^1=(1,1,1,1,-4,0,0,0); γ^2=(1,1,1,1,1,-5,0,0)	$SU(5) \times SU(4) \times U(1) \times SU(6)' \times SU(3)' \times SU(2)'$	n=2

Table 2.1: Embeddings of the spin connection in the gauge degrees of freedom

0	1	1	0	0	0	0	0	α_1
0	0	-1	1	0	0	0	0	α_2
0	0	0	-1	1	0	0	0	α_3
0	0	0	0	-1	1	0	0	α_4
0	0	0	0	0	-1	-1	0	α_5
0	0	0	0	0	0	1	1	α_6
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	α_7
0	0	0	0	0	0	1	-1	α_8

Table 2.2: A simple root system for E_8

of E_7, generated by the roots $\alpha_2, \cdots, \alpha_8$. One realization is given in table 2.2. For a general $\mathbb{Z}_N$ embedding, the gauge group from the first E_8 would then be $U(1) \times E_7$. For $N = 2$, γ itself is also a root, orthogonal to the others, fulfilling (2.5), and the $U(1)$ is enhanced to an $SU(2)$. On the second E_8, the invariant roots are the roots of $SO(14)$ $\alpha_1, \cdots, \alpha_6, \alpha_8$, and an extra root $(1, -1, 0^6)$ such that the unbroken gauge group is $SO(16)$. The second embedding is obviously analogous, only in this case $N = 3$, therefore $(1, -1, 0^6)$ is not an invariant root anymore. For the left-hand side of the third embedding, the unbroken roots are $\alpha_1, (1, -1, 0^6)$, and the second system, orthogonal to the first $\alpha_3, \cdots \alpha_8$, yielding a perturbative gauge group $SU(3) \times E_6$. On the second E_8, the unbroken roots are $\alpha_1, \cdots, \alpha_7, (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, forming the Dynkin diagram of $SU(9)$. The other examples work out similarly. Note that each of these realizations breaks the original gauge group $E_8 \times E_8$ to a different rank 16 subgroup, containing a nonabelian rank r group G and a $U(1)^{16-r}$ that may be enhanced as in the example above. However, this latter factor is only present in the orbifold limit; for a smooth K3, the gauge group consists merely of G.

$$\begin{pmatrix} 2 & -1 & 0 & & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & & & & 0 \\ 0 & -1 & 2 & -1 & 0 & & & 0 \\ \vdots & 0 & -1 & 2 & -1 & 0 & & 0 \\ & & 0 & -1 & 2 & -1 & 0 & -1 \\ & & & 0 & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & 0 \\ 0 & & \cdots & & -1 & 0 & \cdots & 2 \end{pmatrix} \tag{2.6}$$

Figure 2.1: Cartan matrix of E_8

The perturbative gauge group $G \times G'$ can subsequently be spontaneously broken to a subgroup $G_1 \subset G$ via maximal Higgsing, as explained in section 2.1 within the Calabi-Yau approach of [18]. This subgroup depends on the embedding γ only via its instanton numbers: For the standard embedding with $n = 12$, there are no instantons on the second E_8 and the gauge group E_8' can not be broken at all. For the cases $n = 0, 1, 2$, complete Higgsing is possible. For $n = 3, 4, 6, 8$, there are too few hypermultiplets on E_8' that could be used for Higgsing, and G' can only be broken to a terminal subgroup $G_1 = SU(3), SO(8), E_6, E_7$ [16]. Once again, we consider the standard $\mathbb{Z}_2$ orbifold as an example. The hypermultiplets in the untwisted (θ^0) and twisted (θ^1) sectors transform under $E_7 \times SU(2)$ in the following representations:

$$\begin{aligned} (56, 2) + 4(1, 1) \qquad & (\text{untwisted}, \ \theta^0) \\ 8\left((56, 1) + 4(1, 2)\right) \qquad & (\text{twisted}, \ \theta^1). \end{aligned} \tag{2.7}$$

We can now Higgs the $SU(2)$ giving vevs to three scalars, and we are left with 10 hypermultiplets transforming in the $\mathbf{56}$ of E_7 and 65 singlet hypermultiplets, as advertised in section 2.1. We can then break E_7 further by sequential Higgs mechanism. Since the instanton numbers corresponding to this embedding are $(24, 0)$, we can not break the E_8' from the second E_8 lattice at all. A complete classification of orbifold limits of $K3$ along with their instanton numbers can be found in [16].

$\mathbb{Z}_2$	$(1, -1, 0, 0, 0, 0, 0, 0)$; $(2, 0, 0, 0, 0, 0, 0, 0)$	$SU(2) \times E_7 \times SO(16)'$	$n = 4$
$\mathbb{Z}_3$	$(2, 0, 0, 0, 0, 0, 0, 0)$; $(2, 0, 0, 0, 0, 0, 0, 0)$	$U(1) \times SO(14) \times U(1)' \times SO(14)'$	$n = 0$
$\mathbb{Z}_3$	$(1, 1, -2, 0, 0, 0, 0, 0)$; $(-2, 1, 1, 1, 1, 1, 2, 1)$	$SU(3) \times E_6 \times SU(9)'$	$n = 3$
$\mathbb{Z}_4$	$(3, -1, 0, 0, 0, 0, 0, 0)$; $(0, 0, 0, 0, 0, 0, 0, 0)$	$SU(2) \times U(1) \times SO(12) \times E_8'$	$n = 12$
$\mathbb{Z}_6$	$(3, -1, -1, -1, -1, -1, 1, 1)$; $(3, -3, 2, 0, 0, 0, 0, 0)$	$U(1)^2 \times SU(7) \times U(1)' \times SU(2)'^2 \times SO(10)'$	$n = 2$

Table 2.3: Other $\mathbb{Z}_N$ embeddings of the spin connection

2.3 Chains of dual models and the sequential Higgs mechanism

Once one has chosen a modular invariant embedding of $SU(N)$ bundles, and maximally Higgsed the gauge group on the E_8 lattice where the embedding has the lower instanton number, one can perform a cascade breaking on the remaining gauge group along the chain $E_8 \to E_7 \to E_6 \to SO(10) \to SU(5) \to SU(4) \to SU(3) \to SU(2) \to$ (nothing). For the example of the standard $\mathbb{Z}_2$ orbifold, this goes as follows.
Starting with the (65,19) model with $E_7 \times E_8$ symmetry remaining after the gauge embedding, one can move to a point in moduli space where the E_7 gauge symmetry is restored. Under the maximal subgroup $E_6 \times U(1) \in E_7$, the **56** of E_7 decomposes as $\mathbf{56} = \mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1}$. At this point, there are 10 **56**, therefore 20 E_6 singlets charged under the U(1). We now give a generic vev to the adjoint scalars in the unbroken vector multiplets, thereby giving masses to all hypermultiplets charged with respect to E_6, and at the same time breaking E_6 to its maximal Abelian subgroup $U(1)^6$. Using one scalar to Higgs the $U(1)$, we get 19 extra gauge singlet fields: the new spectrum is $(84, 18)$, the second model in the corresponding chain in [8]. We can then move to a point in moduli space where the $U(1)^6$ is enhanced to E_6 and continue this procedure until no gauge symmetry remains on this lattice. In this way, one easily finds a chain of models with characteristics (n_h, n_v) [8]

$$(65, 19), (84, 18), (101, 17), (116, 16), (167, 15), (230, 14), (319, 13), (492, 12) \tag{2.8}$$

The same mechanism can be applied to the other embeddings in table 2.1. For the $\mathbb{Z}_3$ orbifold, $n = 6$, therefore we can maximally Higgs on the second lattice down to E_6. On the first E_8 lattice, we first Higgs down to the rank-reduced subgroup and then start cascade breaking as explained above. The result is a chain $E_6 \to SO(10) \to \cdots \to SU(2) \to 0$ passing through models with characteristics

$$(76, 16), (87, 15), (96, 14), (129, 13), (168, 12), (221, 11), (322, 10). \tag{2.9}$$

For the $\mathbb{Z}_4$ orbifold, $n = 4$, maximal Higgsing leaves an $SO(8)$ on the second lattice and the embedding of the spin connection leaves a rank-reduced subgroup $SU(4)$ on the first. The resulting chain reads

$$(123, 11), (154, 10), (195, 9), (272, 8). \tag{2.10}$$

The $\mathbb{Z}_6$ orbifold in table 2.1, finally, has $n = 2$ and therefore allows for complete Higgsing. The rank-reduced subgroup is $SU(5)$, Higgsed via the chain

$$(118, 8), (139, 7), (162, 6), (191, 5), (244, 4). \tag{2.11}$$

The last four models in each chain have candidate type II duals, i.e. known K3 fibrations with the right Betti numbers. It is interesting to note that on the type-II side, the cascade breaking procedure corresponds precisely to moving between moduli spaces of different Calabi-Yau manifolds. Indeed, as pointed out in [18], this is strikingly similar to the specific type-II process described in [20].

3 Higher derivative couplings for $\mathbb{Z}_n$ orbifolds

We will consider here the $E_8 \times E_8$ formulation of the 10 dimensional heterotic string, where the gauge degrees of freedom are encoded by 16 left-moving bosons, and compactify it

on $K3 \times \mathbb{T}^2$, yielding another two left- and two right-moving bosons. These fields take their values on an even self–dual lattice of signature $(18, 2)$ that will be denoted by $\Gamma^{18,2}$. One can identify $\Gamma^{18,2}$ as obtained from a Euclidean standard lattice by an $SO(18, 2)$ rotation. The moduli space of inequivalent lattices is therefore given by

$$\frac{SO(18,2)}{SO(18) \times SO(2)}. \tag{3.1}$$

This homogeneous space can be parametrized following [4],[17] by

$$u(y) = (\vec{y}, y^+, y^-; 1, -\frac{1}{2}(y, y)), \ y \in \mathbb{C}^{17,1} \tag{3.2}$$

with $y_2 > 0, (y_2, y_2) < 0$ and inner product

$$(x, y) = (\vec{x}, \vec{y}) - x^+ y^- - x^- y^+. \tag{3.3}$$

The right-moving components of a vector in $\Gamma^{18,2}$ with respect to a vector $(\vec{b}, m_-, n_+, m_0, n_0)$ in the fixed Euclidean standard lattice are then denoted by $p_R = p \cdot u(y)$, and we have

$$\frac{p_L^2 - p_R^2}{2} = \frac{1}{2(y_2, y_2)}\big(\vec{b} \cdot \vec{b} + m_- n_+ + m_0 n_0\big), \tag{3.4}$$

$$\frac{p_R^2}{2} = \frac{-1}{2(y_2, y_2)}\big|\vec{b} \cdot \vec{y} + m_+ y^- - n_- y^+ + n_0 + \frac{1}{2} m_0 (y, y)\big|^2, \tag{3.5}$$

The general expression for $F^{(g)}$ is given by [3, 21, 7]

$$F^{(g)} = \frac{1}{Y^{g-1}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{1}{|\eta|^4} \sum_{\text{even}} \frac{i}{\pi} \partial_\tau \left(\frac{\vartheta[^\alpha_\beta](\tau)}{\eta(\tau)} \right) Z_g^{\text{int}}[^\alpha_\beta], \tag{3.6}$$

where

$$Z_g^{\text{int}}[^\alpha_\beta] = \langle : \big(\partial X\big)^{2g} : \rangle = \mathcal{P}_g C_g^{\text{int}}[^\alpha_\beta]. \tag{3.7}$$

$\mathcal{P}_g(q)$ is a one-loop correlation function of the bosonic fields and is given by [22],[3]

$$\mathrm{e}^{-\pi\lambda^2\tau_2} \left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)} \right)^2 = \sum_{g=0}^{\infty} (2\pi\lambda)^{2g} \mathcal{P}_g(q), \tag{3.8}$$

and $C_g^{\text{int}}[^a_b]$ denotes the trace over the (a, b) sector of the internal CFT with an insertion of p_R^{2g-2}, namely

$$\sum_{a,b} c(a, b)(-1)^{2\alpha+2\beta+4\alpha\beta} \frac{\vartheta[^\alpha_\beta]\vartheta[^{\alpha+a}_{\beta+b}]\vartheta[^{\alpha-a}_{\beta-b}]}{\eta^3} \cdot Z_{4,4}[^a_b] \cdot Z^g_{\mathbb{T}^2}[^a_b], \tag{3.9}$$

where $c(a, b)$ are constants ensuring modular invariance.
Note that for g=1, (3.6) is just the unregularized one-loop gravitational threshold correction

$$F^{(1)} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left(\frac{\tau_2}{|\eta|^4} \sum_{\text{even}} \frac{i}{\pi} (-1)^{2\alpha+2\beta+4\alpha\beta} \partial_\tau \left(\frac{\vartheta[^\alpha_\beta](\tau)}{\eta(\tau)} \right) \frac{\widehat{E}_2}{12} C_g^{\text{int}}[^\alpha_\beta] \right). \tag{3.10}$$

The contribution from the bosonic (4,4) blocks reads

$$Z_{4,4}\left[{a \atop b}\right] = 16\frac{\eta^2\bar{\eta}^2}{\vartheta^2\left[{1-a \atop 1-b}\right]\bar{\vartheta}^2\left[{1-a \atop 1-b}\right]} \qquad (a,b) \neq (0,0) \tag{3.11}$$

while the bosons on the $\mathbb{T}^2$ together with the 16 bosons corresponding to the gauge degrees of freedom contribute [17]

$$Z^g_{\mathbb{T}^2}\left[{a \atop b}\right] = \frac{1}{\eta^{18}} e^{-2\pi i a b \gamma^2} \sum_{p\in\Gamma^{18,2}+a\gamma} p_R^{2g-2} e^{2\pi i b\gamma\cdot p} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}}. \tag{3.12}$$

Using

$$\frac{\mathrm{i}}{4\pi}\sum_{(\alpha,\beta)\mathrm{even}} (-1)^{2\alpha+2\alpha+4\alpha\beta}\partial_\tau\left(\frac{\vartheta\left[{\alpha \atop \beta}\right]}{\eta}\right)\frac{\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha+a \atop \beta+b}\right]\vartheta\left[{\alpha-a \atop \beta-b}\right]}{\eta^3}\frac{Z_{4,4}\left[{a \atop b}\right]}{|\eta|^4} = 4\frac{\eta^2}{\bar{\vartheta}\left[{1+a \atop 1+b}\right]\bar{\vartheta}\left[{1-a \atop 1-b}\right]}, \tag{3.13}$$

one can write for (3.6)

$$F^{(g)} = \frac{1}{Y^{g-1}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\tau_2^{2g-1}\mathcal{P}_{2g}(q)\sum_{a,b}\frac{c(a,b)\mathrm{e}^{2\pi\mathrm{i}ab(2-\gamma^2)}}{\eta^{18}\vartheta\left[{1+a \atop 1+b}\right]\vartheta\left[{1-a \atop 1-b}\right]}\sum_{p\in\Gamma^{18,2}+a\gamma} p_R^{2g-2}e^{2\pi i b\gamma\cdot p}q^{\frac{|p_L|^2}{2}}\bar{q}^{\frac{|p_R|^2}{2}}. \tag{3.14}$$

The constants $c(a,b)$ can be determined by the modular invariance constraints [17]

$$\begin{aligned} c(0,b) &= 4\sin^4(\pi b) \\ c(a,b) &= \mathrm{e}^{\pi\mathrm{i}a^2(2-\gamma^2)}c(a,a+b) \\ c(a,b) &= \mathrm{e}^{-2\pi\mathrm{i}ab(2-\gamma^2)}c(b,-a). \end{aligned} \tag{3.15}$$

Introducing the Siegel-Narain theta function with insertion and shifts (see Appendix A)

$$\Theta^g_\Gamma(\tau,\gamma,a,b) = \sum_{p\in\Gamma+a\gamma} p_R^{2g-2}q^{\frac{|p_L|^2}{2}}\bar{q}^{\frac{|p_R|^2}{2}}\mathrm{e}^{\pi\mathrm{i}b\gamma\cdot p}, \tag{3.16}$$

we can rewrite (3.14) as

$$F^{(g)} = \frac{1}{Y^{g-1}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\tau_2^{2g-1}\mathcal{P}_{2g}(q)\sum_{a,b}\frac{c(a,b)\mathrm{e}^{2\pi\mathrm{i}ab(2-\gamma^2)}}{\eta^{18}\vartheta\left[{1+a \atop 1+b}\right]\vartheta\left[{1-a \atop 1-b}\right]}\Theta^g_{\Gamma^{18,2}}(\tau,\gamma,a,b). \tag{3.17}$$

For the special cases of $\mathcal{N}$=2 compactifications with a factorized $\mathbb{T}^2$, the prepotential and $F^{(1)}$ have been shown to be universal, i.e. independent of the specific model [22]. In other words, they are identical for all compactifications on $K3\times\mathbb{T}^2$ with all Wilson lines set to zero. Everything then only depends on the torus moduli. It is easy to see that this also applies to the amplitudes $F^{(g)}$: When we set all Wilson line moduli to zero, the lattice sum obviously factorizes as

$$\sum_{p\in\Gamma^{16,0}+a\gamma} q^{\frac{|p_L|^2}{2}}\mathrm{e}^{2\pi\mathrm{i}bp\cdot\gamma}\sum_{\widehat{p}\in\Gamma^{2,2}} q^{\frac{|\widehat{p}_L|^2}{2}}\bar{q}^{\frac{|\widehat{p}_R|^2}{2}}, \tag{3.18}$$

and we obtain

$$\begin{aligned} F^{(g)}_{\mathrm{0WL}} =& \frac{1}{Y^{g-1}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\tau_2^{2g-1}\mathcal{P}_{2g}(q)\sum_{a,b}\frac{c(a,b)}{\eta^{18}\vartheta[^{1+a}_{1+b}]\vartheta[^{1-a}_{1-b}]}\sum_{p\in\Gamma^{16,0}+a\gamma}q^{\frac{p^2}{2}}\mathrm{e}^{\pi \mathrm{i} g\gamma\cdot p}\Theta^g_{\Gamma^{2,2}}(\tau) \\ =& \int\frac{d^2\tau_2}{\tau_2^2}\tau_2^{2g-1}\mathcal{P}_{2g}\Theta^g_{\Gamma^{2,2}}\frac{1}{\eta^{24}}\Omega, \end{aligned} \tag{3.19}$$

where

$$\Omega = \sum_{a,b}\frac{c(a,b)\eta^6}{\vartheta[^{1+a}_{1+b}]\vartheta[^{1-a}_{1-b}]}\sum_{p\in\Gamma^{16,0}+a\gamma}q^{\frac{p^2}{2}}\mathrm{e}^{\pi \mathrm{i} b\gamma\cdot p}. \tag{3.20}$$

For modular invariance, Ω then has to be a modular form of weight (10,0). Since the spaces of modular forms of even weight $2 < w < 12$ are one-dimensional, Ω has to be proportional to the single generator of weight 10 holomorphic modular forms E_4E_6. Indeed, one finds easily

$$\Omega = \sum_{a,b}\frac{\eta^6}{\vartheta[^{1+a}_{1+b}]\vartheta[^{1-a}_{1-b}]}\sum_{A,B\in\{0,1\}}\prod_{i=1}^{8}\vartheta[^{A+a\gamma_i}_{B+b\gamma_i}] \tag{3.21}$$

which can be checked to be $-E_4E_6$. An abstract proof of this identity based on 6d anomaly cancellation can be found in [23]. We thus find that (3.19) yields precisely the expression for the STU-model without Wilson line moduli given in [5]. This universality property is related to the structure of the elliptic genus [22, 24].
We will now consider the nontrivial case with non-vanishing Wilson lines. The lattice sum does not factorize completely anymore. However, it should factorize partly, into a preserved and a Higgsed part. Indeed, it turns out that one can now write $F^{(g)}$ as

$$F^{(g)} = \frac{1}{Y^{g-1}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\tau_2^{2g-2}\bar{\mathcal{P}}_{2g}(q)\sum_{a,b}\frac{c(a,b)\mathrm{e}^{2\pi \mathrm{i} ab(2-\gamma^2)}}{\eta^{18}\vartheta[^{1+a}_{1+b}]\vartheta[^{1-a}_{1-b}]}\sum_J\bar{\Theta}^g_{J,k}(\tau)\Phi^k_J[^a_b](q) \tag{3.22}$$

with

$$\bar{\Theta}^g_{J,k} = \sum_{p\in\Gamma^{k+2,2}_J}\bar{p}_R^{2g-2}q^{\frac{|p_L|^2}{2}}\bar{q}^{\frac{|p_R|^2}{2}}, \tag{3.23}$$

where $\Gamma^{k+2,2}_J$ denotes the conjugacy class J inside the lattice $\Gamma^{k+2,2}$, and $\Phi^k_J[^a_b](q)$ is a sum over theta functions that will be determined in the following section. Note that (3.22) is manifestly automorphic under the T-duality group $SO(2+k,2;\mathbb{Z})$, since it has the structure of a Borcherds' type one-loop integral [13].

4 Wilson lines: Splitting the lattice

4.1 Decompositions of the E_8 lattice

Recall from section 2.3 that the sequential Higgs mechanism is realized by moving along specific branches of moduli space, away from the generic point. This corresponds to imposing constraints on the Wilson line moduli, such that at each step in the chain, the

number of free Wilson line moduli is reduced by one. The lattice then splits non-trivially into a Higgsed part with $p \cdot y = 0$ and a part depending on the remaining unconstrained moduli from Wilson lines and the torus.

First of all, we will determine how the lattice sum of E_8 behaves under decomposition into the maximal subgroups involved in the cascade breaking. Consider the Dynkin diagram of E_8 (Fig. 4.1) and the simple root system given in table 2.2. In all the figures, crosses correspond to Higgsed generators of the group, while the generators remaining in the Coulomb phase due to Wilson lines are shown as circles. Note that as can be seen

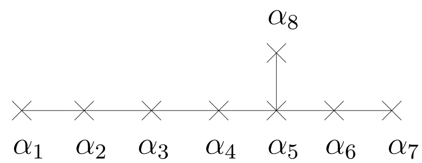

Figure 4.1: E_8 Higgsed completely (no Wilson lines)

from the labeling of the Dynkin diagram, the subgroup E_7 of E_8 is spanned by $\alpha_2, \cdots, \alpha_8$, E_6 by $\alpha_3, \cdots, \alpha_8$, $E_5 = SO(10)$ by $\alpha_4, \cdots, \alpha_8$, and so on for $SU(5), SU(4), SU(3), SU(2)$. We denote the simple roots of the second E_8 by α'_i.

We can now turn on one Wilson line, $y \sim \alpha_1$. On the other hand, turning on seven Wilson line moduli can be encoded in the constraint $\alpha_1 \cdot y = 0$. Both cases result in a split of the lattice sum of E_8 into

$$\begin{aligned}
\sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}} &= \sum_{n_i\in\mathbb{Z}} q^{n_1^2+\cdots+n_8^2-n_1n_2-n_2n_3-n_3n_4-n_4n_5-n_5n_6-n_5n_8-n_6n_7} \\
&= \sum_{n_i\in\mathbb{Z}} q^{(n_1-\frac{n_2}{2})^2+\frac{3}{4}n_2^2+n_3^2+\cdots+n_8^2-n_2n_3-\cdots-n_6n_7} \\
&= \sum_{j=0,1}\sum_{n_1} q^{(n_1-\frac{j}{2})^2} \sum_{n_2,\cdots n_8\in\mathbb{Z}} q^{\frac{3}{4}(2n_2-j)^2+n_3^2+\cdots+n_8^2-(2n_2-j)n_3-\cdots-n_6n_7} \\
&= \sum_{j=0,1} \vartheta\left[{}^{j/2}_{\;0}\right](2\cdot) \sum_{n_2,\cdots,n_8} q^{\frac{3}{4}(2n_2-j)^2+n_3^2\cdots+n_8^2-(2n_2-j)n_3-\cdots-n_6n_7}.
\end{aligned} \tag{4.1}$$

Here and in the following, arguments $(m\cdot)$ stand for $m{\cdot}\tau$, see appendix A. The second sum in the last line is nothing else than the sum over the conjugacy class of E_7 corresponding to $(\alpha_1, p) = j$:

$$\begin{aligned}
(\alpha_1, p) = 2n_1 - n_2 \overset{!}{=} j \qquad &\Rightarrow n_2 = 2n_1 - j \\
&\Rightarrow p = n_1\alpha_1 + (2n_1-j)\alpha_2 + n_3\alpha_3 + \cdots + n_8\alpha_8, \\
p^2 = \frac{3}{2}&(2n_1-j)^2 + \frac{j^2}{2} + 2n_3^2 - 2n_3(2n_1-j) - \cdots
\end{aligned} \tag{4.2}$$

and therefore

$$q^{\frac{j^2}{4}} \sum_{n_2,\cdots,n_8} q^{\frac{3}{4}(2n_2-j)^2+n_3^2\cdots+n_8^2-(2n_2-j)n_3-\cdots-n_7n_8} = \sum_{(p,\alpha_1)=j}^{E_8} q^{\frac{p^2}{2}} = q^{\frac{j^2}{4}} \sum_{E_7^{(1)}} q^{\frac{p^2}{2}}. \tag{4.3}$$

We can also express the above in terms of theta functions. Rewriting the exponent in the second sum in the last line of (4.1) as a sum over p with $(p,\alpha_1)=0$ i.e. as

$$\begin{aligned}
p &= (n_1-\frac{j}{2})\alpha_1+(2n_1-j)\alpha_2+n_3\alpha_3+\cdots n_8\alpha_8\\
&= (-\frac{n_7}{2}, n_1-\frac{j}{2}-\frac{n_7}{2}, -n_1+\frac{j}{2}+\frac{n_7}{2}, 2n_1-j-n_3+\frac{n_7}{2}, n_3-n_4+\frac{n_7}{2},\\
&\qquad n_4-n_5+\frac{n_7}{2}, -n_5+n_6-\frac{n_7}{2}+n_8, n_6-\frac{n_7}{2}-n_8),
\end{aligned}\tag{4.4}$$

we can write this sum as

$$\begin{aligned}
&\sum_{n_2,\cdots,n_8} q^{\frac{3}{4}(2n_2-j)^2+n_3^2\cdots+n_8^2-(2n_2-j)n_3-\cdots-n_7n_8} = \sum_{p\in E_7^{(1)}} q^{\frac{p^2}{2}} = \sum_{\substack{p\in\Gamma_{E_8}-j\frac{\alpha_1}{2}\\(p,\alpha_1)=0}} q^{\frac{p^2}{2}}\\
&= \sum_{\substack{N_1,N_3,\cdots N_8\\ N_3+\cdots+N_8=j \mod 2\\ a=0,1}} q^{(N_1-\frac{j}{2}-\frac{a}{2})^2} q^{\frac{1}{2}\left((N_3-\frac{a}{2})^2+\cdots+(N_8-\frac{a}{2})^2\right)}\\
&= \sum_{\substack{N_1,\cdots N_8\in\mathbb{Z}\\ a=0,1\\ b=0,1}} q^{(N_1-\frac{j}{2}-\frac{a}{2})^2} q^{\frac{1}{2}\left((N_3-\frac{a}{2})^2+\cdots+(N_8-\frac{a}{2})^2\right)}(-1)^{b(N_3+\cdots+N_8-j)}\\
&= \sum_{a,b\in\{0,1\}} \vartheta[{}^{a/2+j/2}_{\ \ 0}](2\cdot)\vartheta[{}^{a/2}_{b/2}]^6(-1)^{jb}.
\end{aligned}\tag{4.5}$$

We thus have decomposed the E_8-lattice according to $P_{E_8}\to P_{E_7^{(0)}}P_{A_1^{(0)}}+P_{E_7^{(1)}}P_{A_1^{(1)}}$, as shown in figure 4.2. This split has already been constructed in [12]. Indeed (4.1) is completely equivalent to the hatting procedure for Jacobi theta functions developed in [12] for this particular split.

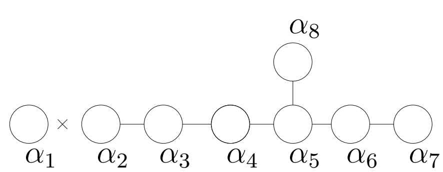

Figure 4.2: $E_8\to E_7\times SU(2)$

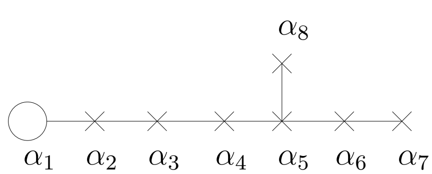

Figure 4.3: E_8 with 1 Wilson line

The same procedure applies when we split the lattice in other maximal subgroups.

Namely, we can decompose with respect to $E_8 \supset E_6 \times SU(3)$:

$$\begin{aligned}
\sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}} &= \sum_{j_2=0,1,2} \sum_{\substack{n_1,n_2\in\mathbb{Z}\\ j_1\in\{0,1\}}} q^{(n_1-\frac{j_1}{2})^2+3(n_2+\frac{j_1}{2}-\frac{j_2}{3})^2} \sum_{n_3,\cdots,n_8\in\mathbb{Z}} q^{\frac{2}{3}(3n_3-j_2)^2+n_4^2+\cdots+n_8^2-(3n_3-j_2)n_4-\cdots-n_6n_7} \\
&= \sum_{\substack{j_1=0,1\\ j_2=0,1,2}} \vartheta[{}^{j_1/2}_{0}](2\cdot)\vartheta[{}^{j_1/2+j_2/3}_{0}](6\cdot) \sum_{a,b\in\{0,1\}} \vartheta[{}^{a/2+j_2/3}_{b/2}](3\cdot)\vartheta[{}^{a/2}_{b/2}]^5(-1)^{b\cdot j_2} \\
&= P_{E_6^{(0)}}\cdot P_{A_2^{(0)}} + 2P_{E_6^{(1)}}\cdot P_{A_2^{(1)}},
\end{aligned} \tag{4.6}$$

The last relation in (4.6) follows from

$$\sum_{n_3,\cdots,n_8\in\mathbb{Z}} q^{6(n_3-\frac{j}{3})^2+n_4^2+\cdots+n_8^2-n_3n_4-\cdots-n_6n_7} = q^{-\frac{j^2}{3}} \sum_{\substack{p\in\Gamma_{E_8}\\ (p,\alpha_1)=0\\ (p,\alpha_2)=j}} q^{\frac{p^2}{2}} = \sum_{E_6^{(j)}} q^{\frac{p^2}{2}}, \tag{4.7}$$

and from the fact that $E_6^{(j=1)}$ and $E_6^{j=2}$ are equivalent. This case corresponds to 2 respectively 6 Wilson lines.

Analogously, we have lattice decompositions with respect to $E_8 \supset SO(10) \times SU(4)$ (3 or 5 Wilson lines)

$$\begin{aligned}
&\sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}} \\
&= \sum_{j_3=0,1,2,3} \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}\\ j_1\in\{0,1\}\\ j_2\in\{0,1,2\}}} q^{(n_1-\frac{j_1}{2})^2+3(n_2+\frac{j_1}{2}-\frac{j_2}{3})^2+6(n_3+\frac{j_2}{3}-\frac{j_3}{4})^2} \sum_{n_4,\cdots,n_8\in\mathbb{Z}} q^{\frac{3}{8}(4n_4-j_3)^2+\cdots+n_8^2-(4n_4-j_3)n_5-\cdots-n_6n_7} \\
&= \sum_{j_3=0,1,2,3} \sum_{\substack{j_1=0,1\\ j_2=0,1,2}} \vartheta[{}^{j_1/2}_{0}](2\cdot)\vartheta[{}^{j_2/3-j_1/2}_{0}](6\cdot)\vartheta[{}^{j_3/4-j_2/3}_{0}](12\cdot) \sum_{a,b\in\{0,1\}} \vartheta[{}^{a/2+j_3/4}_{0}](4\cdot)\vartheta[{}^{a/2}_{b/2}]^4(-1)^{b\cdot j_3} \\
&= P_{D_5^{(0)}}\cdot P_{A_3^{(0)}} + 2P_{D_5^{(1)}}\cdot P_{A_3^{(1)}} + P_{D_5^{(2)}}\cdot P_{A_3^{(2)}},
\end{aligned} \tag{4.8}$$

and for $E_8 \supset SU(5) \times SU(5)$ (4 Wilson lines)

$$\begin{aligned}
\sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}} &= \sum_{j_4=0,\cdots,4} \sum_{\substack{j_1=0,1\\ j_2=0,1,2\\ j_3=0,\cdots,3}} \vartheta[{}^{j_1/2}_{0}](2\cdot)\vartheta[{}^{j_2/3-j_1/2}_{0}](6\cdot)\vartheta[{}^{j_3/4-j_2/3}_{0}](12\cdot)\vartheta[{}^{j_4/5-j_3/4}_{0}](20\cdot)\cdot \\
&\qquad \cdot \sum_{a,B\in\{0,1\}} \vartheta[{}^{a/2+j_4/5}_{B/2}](5\cdot)\vartheta[{}^{a/2}_{B/2}]^3(-1)^{B\cdot j_4} \\
&= P_{A_4^{(0)}}\cdot P_{A_4^{(0)}} + 2P_{A_4^{(1)}}\cdot P_{A_4^{(1)}} + 2P_{A_4^{(2)}}\cdot P_{A_4^{(2)}}.
\end{aligned} \tag{4.9}$$

Note, however, that there are many other ways to decompose the lattice under other maximal subgroups. As an example, we can decompose $E_8 \to SO(14) \times SU(2)$ as shown in figure 4.4:

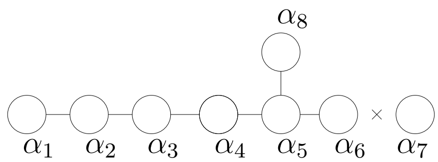

Figure 4.4: The split $E_8 \to SO(14) \times SU(2)$

$$\sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}} = \sum_{j=0,1}\sum_{n_7} q^{(n_7-\frac{j}{2})^2} \sum_{n_1,\cdots,n_6,n_8} q^{\frac{3}{4}(2n_6-j)^2+n_8^2+n_5^2\cdots+n_1^2-(2n_6-j)n_5-n_5n_8\cdots-n_7n_8}. \qquad (4.10)$$

Denoting the lattice sum $\sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}}$ by $f(\tau)$, the splittings (4.1)-(4.9) labeled by the lower number of Wilson lines $k = 1,\cdots,4$ can be cast into the general form

$$f(\tau) = f_0^k\theta_0^{(8-k)} + \cdots f_k^k\theta_k^{(8-k)}, \qquad (4.11)$$

where

$$\theta_J^{(k)} := \sum_{\substack{j_1=0,1\\ \vdots \\ j_{k-1}=0,\cdots k-1}} \vartheta[{\textstyle\frac{j_1}{2} \atop 0}](2\cdot)\vartheta[{\textstyle\frac{j_2}{3}-\frac{j_1}{2} \atop 0}](6\cdot)\cdots\vartheta[{\textstyle\frac{j_{k-1}}{k}-\frac{j_{k-2}}{k-1} \atop 0}]((k-1)\cdot k)\vartheta[{\textstyle\frac{J}{(k+1)}-\frac{j_{k-1}}{k} \atop 0}](k\cdot(k+1)), \qquad (4.12)$$

and

$$f_J^k = q^{-\frac{kJ^2}{2(k+1)}} \sum_{\substack{p\in\Gamma_{E_8}\\ (p,\alpha_1)=\cdots=(p,\alpha_{k-1})=0\\ (p,\alpha_k)=J}} q^{\frac{p^2}{2}}. \qquad (4.13)$$

For the chains of models in [8], we find the explicit expressions

$$f_J^k = \sum_{a,b=0,1} \vartheta[{a/2+J/(k+1) \atop b/2}]((k+1)\cdot)\vartheta[{a/2 \atop b/2}]^{(7-k)}(-1)^{b\cdot J} \qquad (4.14)$$

for k even and

$$f_J^k = \sum_{a,b=0,1} \vartheta[{a/2+J/(k+1) \atop 0}]((k+1)\cdot)\vartheta[{a/2 \atop b/2}]^{(7-k)}(-1)^{b\cdot J} \qquad (4.15)$$

for k odd.
We can write down the same decompositions including the shifts due to the orbifold embedding. In the chains of models in [8], the shifts are of the form $\gamma = (\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m)$ and thus deform p to $p + a\gamma = (n_1+a)\alpha_1 + (n_2+2a)\alpha_2 + \cdots + (n_m + m\cdot a)\alpha_j$. Therefore, $\theta_J^{(k)}$ gets deformed to

$$\theta_{J,\gamma}^{(k)}[{a \atop b}](q) = \sum_{\substack{j_1=0,1\\ \vdots \\ j_{k-1}=0,\cdots k-1}} \vartheta[{\textstyle\frac{j_1}{2} \atop 0}](2\cdot)\vartheta[{\textstyle\frac{j_2}{3}-\frac{j_1}{2} \atop 0}](6\cdot)\cdots\vartheta[{\textstyle\frac{j_m}{(m+1)}-\frac{j_{m-1}}{m}-m\cdot a \atop -m(m+1)b}](m\cdot(m+1))\cdots\vartheta[{\textstyle\frac{J}{(k+1)}-\frac{j_{k-1}}{k} \atop 0}](k\cdot(k+1)). \qquad (4.16)$$

Similar realizations exist for other types of shifts. On the part of the lattice denoted by f^k_J, it is more convenient to write in an orthogonal basis $\gamma = (\gamma_1, \cdots, \gamma_{7-k}, 0, \cdots, 0)$ and we get for f^k_J with k even

$$f^k_{J,\gamma}[{}^a_b] = \sum_{A,B=0,1} \mathrm{e}^{-\pi \mathrm{i} \sum_i \gamma_i B a} \vartheta[{}^{A/2+J/(k+1)}_{B/2}]((k+1)\cdot) \prod_{i=1}^{7-k} \vartheta[{}^{A/2+a\gamma_i}_{B/2+b\gamma_i}](-1)^{B\cdot J}, \tag{4.17}$$

respectively for k odd,

$$f^k_{J,\gamma}[{}^a_b] = \sum_{A,B=0,1} \mathrm{e}^{-\pi \mathrm{i} \sum_i \gamma_i B a} \vartheta[{}^{A/2+J/(k+1)}_{0}]((k+1)\cdot) \prod_{i=1}^{7-k} \vartheta[{}^{A/2+a\gamma_i}_{B/2+b\gamma_i}](-1)^{B\cdot J}. \tag{4.18}$$

Cases with more than $7-k$ non-vanishing entries in γ have to be considered separately, see section 4.2.
The lattice splits derived above are the main ingredients for computing the $F^{(g)}$ in models with Wilson lines. Indeed, turning on one Wilson line in the chains of [8] corresponds to preserving a $U(1)$ that can be enhanced to an $SU(2)$ while Higgsing an E_7, and will therefore be reflected by a split as in (4.1). On the other hand, turning on seven Wilson lines Higgses an $SU(2)$ while preserving a $U(1)^7$ that can be enhanced to E_7 and therefore corresponds to the same split with sides exchanged, or equivalently: the same modified Dynkin diagram (Fig. 4.2) with circles replaced by crosses. Similarly, (4.6) corresponds to 2, respectively 6 and (4.8) to 3, respectively 5 Wilson lines. For 4 Wilson lines, one can choose to Higgs either side of the lattice.

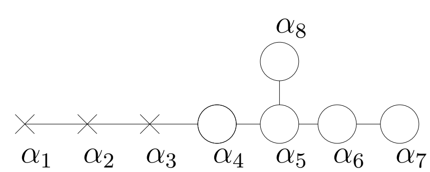

Figure 4.5: E_8 with 5 Wilson lines

4.2 Moduli dependence

We can now use the above to decompose the full lattice sum with torus moduli, Wilson moduli, shifts and insertions. Note that when the vector of Wilson line moduli y is *not* orthogonal to the shifts, i.e. $\gamma \cdot y \neq 0$, we turn on Wilson line moduli corresponding to the part of the gauge group only present in the orbifold limit. This results in freezing the vector moduli at that special point of moduli space, and the degeneracy of vacua gets lifted: The couplings corresponding to equivalent embeddings with different N can

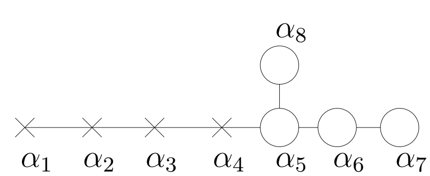

Figure 4.6: E_8 with 4 Wilson lines

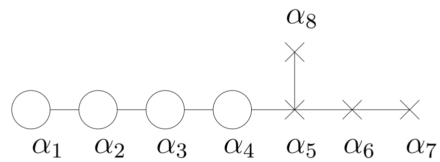

Figure 4.7: E_8 with 4 Wilson lines, alternative split

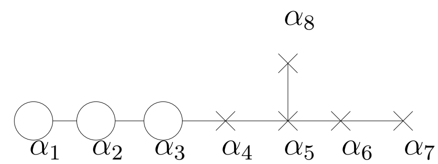

Figure 4.8: E_8 with 3 Wilson lines

be different [17].
We therefore impose here $\gamma \cdot y = 0$, restricting the Wilson lines to the part of the lattice orthogonal to the shift. We have to distinguish the cases of less than four Wilson lines from those with four and more. In the latter, $\gamma \cdot y = 0$ is automatically fulfilled for the shifts given in table 2.1, as the Wilson lines are active on the right-hand side of the Dynkin diagram while the shifts act on the left. If we turn on less than four Wilson lines, those are active on the left-hand side of the diagram, as explained in section 4.1. This means that we have to choose the shift such that it does not interfere with the Wilson lines, and in such a way that it preserves the part of the diagram where the Wilson lines are active. For the $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_4$ embeddings on the first E_8 lattice (see table 2.1), it is sufficient to move the shift to the other end of the diagram, redefining $\gamma^1_{\mathbb{Z}_2} \to \gamma'^1_{\mathbb{Z}_2} = (0^6, -1, 1), \gamma^1_{\mathbb{Z}_3} \to \gamma'^1_{\mathbb{Z}_3} = (0^5, -2, 1, 1), \gamma_{\mathbb{Z}_4} \to \gamma'^1_{\mathbb{Z}_4} = (0^4, -3, 1, 1, 1)$. In the case of the $\mathbb{Z}_6$ orbifold, this does the trick for one and two Wilson lines, but if we turn on a third one, it is not orthogonal to $\gamma'^1_{\mathbb{Z}_6}$ anymore. However, we can choose the equivalent embedding $\gamma'^1 = (2, 2, 2, 2, 2, 0^3)$, orthogonal to $y \in \text{span}(\alpha_1, \alpha_2, \alpha_3)$. In this case, this is also a valid choice for zero, one and two Wilson lines. The Wilson lines on the second E_8, unchanged throughout the sequential Higgs mechanisms, work out similarly. Only the $\mathbb{Z}_4$ orbifold is slightly more delicate, as the Wilson lines corresponding to maximal Higgsing on the second E_8 preserve an $SO(8)$, and therefore act in the center of the diagram. The combination of theta functions corresponding to the Higgsed lattice can however be determined using (4.13).

For one Wilson line, we thus write

$$\sum_{p\in\Gamma^{18,2}+a\gamma} p_R^{(2g-2)} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}} \mathrm{e}^{2\pi \mathrm{i} b\gamma\cdot p} = \sum_{p\in\Gamma^{18,2}+a\gamma} (p\cdot u(y))^{(2g-2)} q^{\frac{p^2}{2}} |q|^{(p\cdot u(y))^2} \mathrm{e}^{2\pi \mathrm{i} b\gamma\cdot p}$$

$$= \sum_{J=0,1} \sum_{\substack{A,B\in\{0,1\}\\ \alpha,\beta\in\{0,1\}}} \mathrm{e}^{-\pi \mathrm{i}\sum_i \gamma_i' Ba} \left(\prod_{i=3}^{8} \vartheta\big[{}^{A/2+a\gamma_i'}_{B/2+b\gamma_i'}\big]\right) \vartheta\big[{}^{A/2+J/2}_{0}\big](2\cdot)(-1)^{BJ}$$

$$\cdot\, \mathrm{e}^{-\pi \mathrm{i} a\sum_{i=9}^{16}\gamma_i\beta} \left(\prod_{j=9}^{16} \vartheta\big[{}^{\alpha/2+a\gamma_j}_{\beta/2+b\gamma_j}\big]\right) \cdot \sum_{n_1,n_\pm,m_\pm} (p\cdot u(y))^{2g-2} q^{(n_1-\frac{J}{2})^2 - m_+ n_- + n_0 m_0} |q|^{(p\cdot u(y))^2}$$

$$= \sum_J f_J^1\big[{}^a_b\big](q)\bar{\Theta}^g_{J,1}(q,y), \tag{4.19}$$

where $\Theta^g_{J,k}(q,y)$ is defined in (3.23), and

$$f_J^1\big[{}^a_b\big](q) = \sum_{\substack{A,B\in\{0,1\}\\ \alpha,\beta\in\{0,1\}}} \mathrm{e}^{-\pi \mathrm{i} a\sum_{i=3}^{8}\gamma_i' B} \left(\prod_{i=3}^{8} \vartheta\big[{}^{A/2+a\gamma_i'}_{B/2+b\gamma_i'}\big]\right) \vartheta\big[{}^{A/2+J/2}_{0}\big](2\cdot)(-1)^{BJ}$$
$$\cdot\, \mathrm{e}^{-\pi \mathrm{i} a\sum_{i=9}^{16}\gamma_i\beta} \left(\prod_{j=9}^{16} \vartheta\big[{}^{\alpha/2+a\gamma_j}_{\beta/2+b\gamma_j}\big]\right). \tag{4.20}$$

This is nothing else than (4.18) applied to the whole lattice of two E_8 and the torus, and including the shifts. Analogously, we get for $k \leq 4$ Wilson lines

$$\sum_{p\in\Gamma^{18,2}+a\gamma} p_R^{(2g-2)} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}} \mathrm{e}^{2\pi \mathrm{i} b\gamma\cdot p} = \sum_J f_J^k\big[{}^a_b\big](q)\bar{\Theta}^g_{J,k}(q,y), \tag{4.21}$$

where for k=3

$$f_J^3\big[{}^a_b\big](q) = \sum_{\substack{A,B\in\{0,1\}\\ \alpha,\beta\in\{0,1\}}} \mathrm{e}^{-\pi \mathrm{i} a\sum_{i=5}^{8}\gamma_i' B} \left(\prod_{i=5}^{8} \vartheta\big[{}^{A/2+a\gamma_i'}_{B/2+b\gamma_i'}\big]\right) \vartheta\big[{}^{A/2+J/4}_{0}\big](4\cdot)(-1)^{BJ}$$
$$\mathrm{e}^{-\pi \mathrm{i} a\sum_{i=9}^{16}\gamma_i\beta} \left(\prod_{j=9}^{16} \vartheta\big[{}^{\alpha/2+a\gamma_j}_{\beta/2+b\gamma_j}\big]\right), \tag{4.22}$$

and for $k = 2$ or $k = 4$ Wilson lines, using (4.17),

$$f_J^k\big[{}^a_b\big](q) = \sum_{\substack{A,B\in\{0,1\}\\ \alpha,\beta\in\{0,1\}}} \mathrm{e}^{-\pi \mathrm{i} a\sum_{i=k+2}^{8}\gamma_i B} \left(\prod_{i=k+2}^{8} \vartheta\big[{}^{A/2+a\gamma_i'}_{B/2+b\gamma_i'}\big]\right) \vartheta\big[{}^{A/2+J/(k+1)}_{B/2}\big]((k+1)\cdot)(-1)^{BJ}$$
$$\mathrm{e}^{-\pi \mathrm{i} a\sum_{i=9}^{16}\gamma_i\beta} \left(\prod_{j=9}^{16} \vartheta\big[{}^{\alpha/2+a\gamma_j}_{\beta/2+b\gamma_j}\big]\right). \tag{4.23}$$

When more than four Wilson lines are turned on ($k \geq 4$), we decompose analogously as

$$\sum_{p\in\Gamma^{18,2}+a\gamma} p_R^{(2g-2)} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}} \mathrm{e}^{2\pi \mathrm{i} b\gamma\cdot p} = \sum_J \theta_J^k[{}^a_b](q)\bar{\Theta}^g_{J,k}(q,y), \tag{4.24}$$

where $\theta_J^k[{}^a_b](q)$ is (4.16), supplemented by the contribution from the second E_8 lattice. Any other split for any number of Wilson lines fulfilling the constraint $\gamma \cdot y = 0$ can be realized similarly. In the above, we have assumed that the second E_8 lattice is Higgsed completely, without any Wilson lines. If this is not the case, as for example for the $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_4$ models in [8], the second lattice also has to be split according to the above prescription.

Note that these splits describe a "generalized hatting procedure" analogous to the 1-Wilson line case analyzed in [12] for generalized Jacobi forms. In the 1 Wilson line $STUV$ model, the relevant forms are standard Jacobi forms

$$f(\tau, V) = \sum_{\substack{n\geq 0\\ l\in\mathbb{Z}}} c(4n-l^2)q^n r^l \tag{4.25}$$

with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}, r = \mathrm{e}^{2\pi V}$, admitting a decomposition

$$f(\tau, V) = f_{ev}(\tau)\theta_{ev}(\tau, V) + f_{odd}(\tau)\theta_{odd}(\tau, V), \tag{4.26}$$

where $\theta_{ev} = \theta_3(2\tau, 2V)$, $\theta_{odd} = \theta_2(2\tau, 2V)$. The effect of turning on a Wilson line can be described by replacing $f(\tau, V)$ by its hatted counterpart [12]

$$\hat{f}(\tau, V) = f_{ev}(\tau) + f_{odd}(\tau) \tag{4.27}$$

In the generic, k Wilson line case considered here, we decompose the lattice sum as in (4.11).

When $k \leq 4$, the "generalized hatting" due to the Wilson lines is

$$\hat{f}[{}^a_b](\tau, V_1, \cdots V_k) = f_0^k[{}^a_b](\tau) + \cdots f_k^k[{}^a_b](\tau), \tag{4.28}$$

where f_J^k and f_{k+1-J}^k are equivalent. When $k \geq 4$, we have to keep the other part of the split lattice. This yields the "complementary hatting"

$$\breve{f}(\tau, V_1, \cdots V_n) = \theta_0^{8-k}[{}^a_b](\tau) + \cdots \theta_k^{8-k}[{}^a_b](\tau), \tag{4.29}$$

with $\theta_J^{8-k} = \theta_{k+1-J}^{8-k}$.

4.3 Computation of $F^{(g)}$

In the following, we will denote the number of Wilson lines by k and write the split lattice sum as

$$\sum_J \Phi_J^k[{}^a_b](q)\bar{\Theta}^g_{k,J}(q), \tag{4.30}$$

where $\Phi^k_J[{a \atop b}](q)$ is the function appearing in (3.22) and stands for $f^k_J[{a \atop b}]$ or $\theta^k_J[{a \atop b}](q)$, whichever is applicable. We expand the modular function in the integrand of (3.22) as

$$\mathcal{P}_{2g}(q)\mathcal{F}^k_J(q) := \mathcal{P}_{2g}(q)\sum_{a,b}\frac{c(a,b)\mathrm{e}^{2\pi\mathrm{i}ab(2-\gamma^2)}}{\eta^{18}\vartheta[{1+a \atop 1+b}]\vartheta[{1-a \atop 1-b}]}\Phi^k_J[{a \atop b}](q) = \sum_{n\in\mathbb{Q}_J} c^k_{g,J}(n)q^n, \tag{4.31}$$

where $\mathbb{Q}_J$ denotes the subset of $\mathbb{Q}$ containing the powers of q appearing in the conjugacy class J. Since different conjugacy classes correspond to different rational powers of q, we can sum over J without loss of information and write

$$\sum_{n\in\mathbb{Q}} c^k_g(n)q^n = \sum_J \sum_{n\in\mathbb{Q}_J} c^k_{g,J}(n)q^n. \tag{4.32}$$

We can now evaluate the integral (3.14) using Borcherds' technique of lattice reduction [13] reviewed in appendix B. We choose the reduction vector to lie in the torus part of the lattice, the result is therefore only valid in the chamber of the T, U torus moduli space where the projected reduction vector z_+ is small. The result looks very similar to what was obtained in [5] for the STU-model and can be simplified to read [1] $F^{(g)} = F^{(g)}_{\mathrm{deg}} + F^{(g)}_{\mathrm{nondeg}}$ where

$$F^{(g)}_{\mathrm{deg}} = \frac{(y_2,y_2)8\pi^3}{T_2}\delta_{g,1} + \frac{1}{2(2T_2)^{2g-3}}\sum_{\lambda\in\Gamma^{k,0}}\sum_{l=0}^{g}\mathrm{Li}_{2l-2g+4}(q^{\mathrm{Re}\,(\bar\lambda\cdot\bar y)})c^k_{g-l}(\frac{\lambda^2}{2})\frac{1}{\pi^{2l+3}}(-\frac{T_2^2}{2y_2^2})^l \tag{4.33}$$

$$\begin{aligned} F^{(g)}_{\mathrm{nondeg}} &= \\ &\sum_{l=0}^{g-1}\sum_{C=0}^{\min(l,2g-3-l)}\sum_{r\in\Gamma^{k+1,1}}\binom{2g-l-3}{C}\frac{1}{(l-C)!2^C}\frac{(-\mathrm{Re}\,(r\cdot y))^{l-C}}{(y_2,y_2)^l}c^k_{g-l}(\frac{r^2}{2})\mathrm{Li}_{3-2g+l+C}(\mathrm{e}^{-r\cdot y}) \\ &+ \frac{c^k_1(0)}{2^g(g-1)(y_2,y_2)^{g-1}} + \sum_{l=0}^{g-2}\frac{c^k_{g-l}(0)}{l!(2(y_2,y_2))^l}\zeta(3+2(l-g))\frac{(2g-3-l)!}{(2g-3-2l)!} \end{aligned} \tag{4.34}$$

This can also be compared to the expressions obtained in [17] for genus one. The lattice sum in (4.34) is over the so-called reduced lattice $\Gamma^{k+1,1}$. This is a sublattice of the original lattice $\Gamma^{k+2,2}$, parametrized by (n_0, m_0, b_i).
A highly nontrivial check of the computation is provided by the Euler characteristics of the corresponding Calabi-Yau manifolds, respectively the difference $n_h - n_v$ on the heterotic side. Heterotic-type II duality implies [5] that it should be given by the normalized q^0 coefficient of $\mathcal{F}^k_J$, namely

$$2(n_h - n_v) = \chi(X) = 2\frac{c^k_0(0)}{c^k_0(-1)}. \tag{4.35}$$

One indeed finds precisely the chains of Euler characteristics given in [8], see table 4.1. The corresponding K3-fibrations are listed in table C.1.

[1] see the appendix of [25] for details of the simplification

$\mathbb{Z}_2$	92	132	168	200	304	412	612	960
$\mathbb{Z}_3$		120	144	164	232	312	420	624
$\mathbb{Z}_4$					224	288	372	528
$\mathbb{Z}_6$				220	264	312	372	480

Table 4.1: Euler characteristics χ for the models in [8]

5 Heterotic-type II duality and instanton counting

5.1 Moduli map

In this section, we will determine geometric quantities on the dual Calabi-Yau manifolds on the type II side using the heterotic expressions obtained above.
The heterotic dilaton S gets mapped to the Kähler modulus t_2, therefore heterotic weak coupling regime corresponds to $t_2 \to \infty$. This restricts the instanton numbers accessible to our computation to those classes where the corresponding coefficient l_2 vanishes. The mapping of the remaining heterotic moduli from the Torus and the Wilson lines $(T, U, V_1, \cdots V_k)$ to the Kähler moduli $(t_1, \cdots t_{k+3})$ on the type II side can be determined for models with small number of Kähler moduli comparing the classical pieces of the prepotential [12]. In order to compare with the instanton numbers in [14], we extend the map of [12] to two Wilson lines as follows:

$$\begin{aligned} T &\to t_1 + 2t_4 + 3t_5 \\ U &\to t_1 + t_3 + 2t_4 + 3t_5 \\ V_1 &\to t_4 \\ V_2 &\to t_5 \end{aligned} \tag{5.1}$$

implying that the numbers (n_0, m_0, b_i) in (3.5) map to the numbers l_i on the type II side as

$$\begin{array}{lll} l_1 &= n_0 + m_0 & l_4 = 2(n_0 + m_0) + b_1 \\ l_2 &= 0 & l_5 = 3(n_0 + m_0) + b_2 \\ l_3 &= n_0. & \end{array} \tag{5.2}$$

For higher numbers of Wilson lines, we cannot conclusively determine the map due to lack of information on the type II side, but it is clear that such a map exists and that it is linear. In order to extract genus g instanton numbers from the expansion (4.31), we

have to specify the norm (p,p). Redefining the indices in (4.1)-(4.9) as

$$\begin{aligned}
&(n_1-\frac{a}{2})^2 && \to \frac{b_1^2}{4}\\
&(n_1-\frac{a}{2})^2+3(n_2+\frac{a}{2}-\frac{b}{3})^2 && \to \frac{b_1^2}{4}+3(\frac{b_1}{2}-\frac{b_2}{3})^2=b_1^2-b_1b_2+\frac{b_2^2}{3}\\
&(n_1-\frac{a}{2})^2+3(n_2+\frac{a}{2}-\frac{b}{3})^2+6(n_2+\frac{b}{3}-\frac{c}{4})^2 && \to \frac{b_1^2}{4}+3(\frac{b_1}{2}-\frac{b_2}{3})^2+6(\frac{b_2}{3}-\frac{b_3}{4})^2\\
& && =b_1^2+b_2^2-b_1b_2-b_2b_3+\frac{3b_3^2}{8},\\
& && \vdots
\end{aligned} \tag{5.3}$$

we find the norms given in table 5.1. We thus have for the instanton numbers

$$\begin{aligned}
&c_k^g(n_0,m_0,b_1,\cdots b_k)=c_k^g(n_0m_0-b_1^2-\cdots-b_{k-1}^2+b_1b_2\cdots b_{k-1}b_k-\frac{kb_k^2}{2(k+1)}), \qquad k\le 4\\
&c_k^g(n_0,m_0,b_{9-k},\cdots b_8)=c_k^g(n_0m_0-\frac{(10-k)b_{9-k}^2}{2(9-k)}-b_{10-k}^2-\cdots-b_8^2+b_{9-k}b_{10-k}+\cdots b_5b_8,\\
&\qquad\qquad k\ge 4,
\end{aligned} \tag{5.4}$$

confirming the conjecture made in [12]. Note that the last b_p determines the conjugacy class.

k	$p_{\rm het}^2$
0	n_0m_0
1	$n_0m_0-\frac{b_1^2}{4}$
2	$n_0m_0-b_1^2+b_1b_2-\frac{b_2^2}{3}$
3	$n_0m_0-b_1^2-b_2^2+b_1b_2+b_2b_3-\frac{3b_3^2}{8}$
4	$n_0m_0-b_1^2-b_2^2-b_3^2+b_1b_2+b_2b_3+b_3b_4-\frac{2b_4^2}{5}$
5	$n_0m_0-\frac{5b_4^2}{8}-b_5^2-b_6^2-b_7^2-b_8^2+b_4b_5+b_5b_6+b_5b_8+b_6b_7+b_7b_8$
6	$n_0m_0-\frac{2b_3^2}{3}-b_4^2-b_5^2-b_6^2-b_7^2-b_8^2+b_3b_4+b_4b_5+b_5b_6+b_5b_8+b_6b_7+b_7b_8$
7	$n_0m_0-\frac{3b_2^2}{4}-b_3^2-b_4^2-b_5^2-b_6^2-b_7^2-b_8^2+b_2b_3+b_3b_4+b_4b_5+b_5b_6+b_5b_8+b_6b_7+b_7b_8$

Table 5.1: The norm $(p_{\rm het},p_{\rm het})_k$ for $k=(0,1,\cdots 7)$ Wilson lines

5.2 Extracting geometric information

The topological couplings $F^{(g)}$ are the free energies of the A-model topological string. They have a geometric interpretation as a sum over instanton sectors,

$$F^{(g)}(t)=\sum_\beta N_{g,\beta}Q^\beta, \tag{5.5}$$

where $Q_i = \mathrm{e}^{-t_i}$, $\beta = \{n_i\}$ in a basis of $H_2(X)$ denotes a homology class, $Q^\beta := \mathrm{e}^{-t_i n_i}$, and $N_{g,\beta}$ are the Gromov-Witten invariants, in general *rational* numbers. With the work of Gopakumar and Vafa [26], a hidden integrality structure of the $N_{g,\beta}$ has been uncovered. The generating functional of the $F^{(g)}$,

$$F(t, g_s) = \sum_{g=0}^{\infty} F^{(g)}(t) g_s^{2g-2}, \tag{5.6}$$

can be written as a generalized index counting BPS states in the corresponding type IIA theory:

$$F(t, g_s) = \sum_{g=0} \sum_{\beta} \sum_{d=1}^{\infty} n_\beta^g \frac{1}{d} \left(2 \sin \frac{d g_s}{2}\right)^{2g-2} Q^{d\beta}, \tag{5.7}$$

where the numbers n_β^g are now *integers* called Gopakumar-Vafa invariants. Since the homology classes β are labeled by lattice vectors p, we write the Gopakumar-Vafa invariants for models with k Wilson lines as $n_g^k(p) \equiv n_g^k(\frac{p^2}{2})$. We also write, in terms of the instanton degrees on the type II side, $n_g^k(l_1, \cdots, l_{k+3})$.

From the structure of the $F^{(g)}$, one can deduce that the coefficients $c_g^k(\frac{p^2}{2})$ appearing in (4.33),(4.34) are related to the Gopakumar-Vafa invariants through

$$\sum_{g \geq 0} n_g^k(p) \left(2 \sin \frac{\lambda}{2}\right)^{2g-2} = \sum_{g \geq 0} c_g^k(\frac{p^2}{2}) \lambda^{2g-2}. \tag{5.8}$$

The Gopakumar-Vafa invariants can be obtained efficiently using the formula [7]

$$\sum_{p \in \mathrm{Pic}(K3)} \sum_{g=0}^{\infty} n_g^k(p) z^g q^{\frac{p^2}{2}} = \sum_J \mathcal{F}_J^k(q) \xi^2(z, q), \tag{5.9}$$

where $\mathcal{F}_J^k(q)$ is defined in (4.31), and

$$\xi(z, q) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n)^2 + z q^n}. \tag{5.10}$$

5.3 Gopakumar-Vafa invariants

Table 5.2- table 5.4 show conjectural GV invariants n_g^k for the K3 fibrations dual to the STU-, the $STUV$-, and the $STUV_1V_2$-model. Similar tables for the other models considered in this work can be found in appendix C, along with a list of the dual pairs of [8].

For comparison with [14], we give the genus 0 instanton numbers in notation $[l_1 \cdots l_{k+3}] = n_0^k(l_1, \cdots l_{k+3})$ for models with one and two Wilson lines in table 5.5, 5.6. We find indeed perfect agreement with [14].

Another nontrivial check is provided by the requirement of consistent truncation: in [14], the authors deduce that the following relations have to hold between instanton numbers with 3,4,and 5 moduli

$$n_0^0(l_1, l_2, l_3) = \sum_x n_0^1(l_1, l_2, l_3, x) \qquad n_0^1(l_1, l_2, l_3, l_4) = \sum_x n_0^2(l_1, l_2, l_3, l_4, x). \tag{5.11}$$

g	$\frac{p^2}{2}=-1$	0	1	2	3	4	5
0	-2	480	282888	17058560	477516780	8606976768	115311621680
1	0	4	-948	-568640	-35818260	-1059654720	-20219488840
2	0	0	-6	1408	856254	55723296	1718262980
3	0	0	0	8	-1860	-1145712	-76777780
4	0	0	0	0	-10	2304	1436990

Table 5.2: $n_g^k(\frac{p^2}{2})$ for $\mathbb{Z}_6$, 0 Wilson lines (STU), dual to $X^{1,1,2,8,12}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{4}$	0	$\frac{3}{4}$	1	$\frac{7}{4}$	2	$\frac{11}{4}$	3
0	-2	56	372	53952	174240	3737736	9234496	110601280	237737328
1	0	0	4	-112	-732	-108240	-350696	-7799632	-19517380
2	0	0	0	0	-6	168	1084	162752	528582
3	0	0	0	0	0	0	8	-224	-1428

Table 5.3: $\mathbb{Z}_6$,1 Wilson line (STUV), dual to $X^{1,1,2,6,10}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{3}$	0	$\frac{2}{3}$	1	$\frac{5}{3}$	2	$\frac{8}{3}$	3
0	-2	30	312	26664	120852	1747986	5685200	49588776	135063180
1	0	0	4	-60	-612	-53508	-243560	-3656196	-12097980
2	0	0	0	0	-6	90	904	80472	367458
3	0	0	0	0	0	0	8	-120	-1188
4	0	0	0	0	0	0	0	0	-10

Table 5.4: $\mathbb{Z}_6$, 2 Wilson lines (STUV_1V_2), dual to $X^{1,1,2,6,8}$

[0001]	56	[1001]	56	[1003]	56	[3014]	174240
[0002]	-2	[1002]	372	[1000]	-2	[1011]	56
[1004]	-2	[2012]	372	[0003]	0	[2013]	53952

Table 5.5: Numbers of rational curves of degree $[l_1, 0, l_2, l_3, l_4]$ on $X^{1,1,2,6,10}$ (dual to $\mathbb{Z}_6$,1 WL)

[00001]	30	[10011]	30	[00002]	0	[10023]	312
[00010]	-2	[10022]	30	[00012]	30	[10010]	-2
[00023]	-2	[20101]	26664	[00011]	30	[20169]	312
[00101]	0	[30141]	0	[00013]	-2	[30144]	30
[30145]	26664	[30146]	120852	[30147]	26664	[30148]	30

Table 5.6: Numbers of rational curves of degree $[l_1, 0, l_3, l_4, l_5]$ on $X^{1,1,2,6,8}$ (dual to $\mathbb{Z}_6$, 2 WL)

Our numbers indeed fulfill this constraint, as for example

$$n_0^2(0,0,0,1,0) + \cdots + n_0^2(0,0,0,1,3) = -2 + 30 + 30 - 2 = 56 = n_0^1(0,0,0,1), \quad (5.12)$$

$$n_0^1(0,0,0,0)+\cdots+n_0^1(0,0,0,4)=-2+56+372+56-2=480=n_0^0(0,0,0), \quad (5.13)$$

and

$$n_0^2(3,0,1,4,0)+\cdots+n_0^2(3,0,1,4,8)=174240=n_0^1(3,0,1,4). \quad (5.14)$$

This relation should also hold at higher genus and for higher numbers of Kähler moduli [6], namely we expect

$$n_g^k(l_1,l_2,\cdots l_{k+3})=\sum_x n_g^{k+1}(l_1,l_2,\cdots l_{k+3},x). \quad (5.15)$$

Indeed, we have for example for truncation from 2 to 1 Wilson line (tables 5.3, 5.4) $4-60-60+4=-112$, $-6+90+90-6=168$, and $90+904+90=1084$. All instanton numbers produced, including those in tables C.2-C.15, fulfill the truncation identities

$$\begin{array}{ll}
n_g^0(1)=2\left(n_g^1(0)+n_g^1(\frac{3}{4})\right)+n_g^1(1) & n_g^0(2)=2\left(n_g^1(-\frac{1}{4})+n_g^1(1)+n_g^1(\frac{7}{4})\right)+n_g^1(2) \\
n_g^1(1)=2\left(n_g^2(-\frac{1}{3})+n_g^2(\frac{2}{3})\right)+n_g^2(1) & n_g^1(2)=2\left(n_g^2(-1)+n_g^2(\frac{2}{3})+n_g^2(\frac{5}{3})\right)+n_g^2(2) \\
n_g^2(1)=2\left(n_g^3(-\frac{1}{2})+n_g^3(\frac{5}{8})\right)+n_g^3(1) & n_g^2(2)=2\left(n_g^3(\frac{1}{2})+n_g^3(\frac{13}{8})\right)+n_g^3(2) \\
n_g^2(\frac{2}{3})=n_g^3(-\frac{3}{8})+n_g^3(0)+n_g^3(\frac{1}{2})+n_g^3(\frac{5}{8}) & n_g^3(0)=n_g^4(-\frac{2}{5})+n_g^4(0).
\end{array} \quad (5.16)$$

Note that these identities hold –as far as we can verify– at general genus and independently of the specific chain, as expected. Again, this provides a non-trivial check of our results.

6 Conclusion

We have shown how to compute higher derivative couplings for general symmetric $\mathbb{Z}_N$, $\mathcal{N}=2$ orbifold compactifications of the heterotic string with any number of Wilson lines. In particular, this provides conjectural instanton numbers for any of the models in the chains of heterotic-type II duals of [8].

Unfortunately, our results can so far only be checked for up to two Wilson lines, since for higher numbers of vector multiplets the type II computation becomes very involved. They do however fulfill nontrivial constraints coming from the geometric transitions on the type II side [14].

Furthermore, a rigorous mathematical framework for computing Gromov-Witten invariants along the fiber of certain K3-fibrations has been established in [29, 30]. With these techniques, one might be able to prove some of our physical predictions for Calabi-Yau manifolds of this type.

The computation is rather general and might be applicable to other models, e.g. to asymmetric orbifolds.

Acknowledgments

I would like to thank M. Mariño for suggesting the topic, many helpful comments and discussions, and a critical reading of the manuscript. I also thank E. Scheidegger for discussions, J. David for comments on related topics, and especially S. Stieberger for very valuable remarks. The major part of this work was carried out under the Marie Curie EST program.

Appendices

A Theta functions

Properties

In our conventions, the theta functions are defined as follows:

$$\vartheta\left[{a \atop b}\right](v|\tau) = \sum_{n \in Z} q^{\frac{1}{2}(n-a)^2} e^{2\pi i (v-b)(n-a)} \tag{A.1}$$

where a, b are rational numbers and $q = e^{2\pi i \tau}$.

They show the following periodicity properties:

$$\vartheta\left[{a+1 \atop b}\right](v|\tau) = \vartheta\left[{a \atop b}\right](v|\tau) \quad , \quad \vartheta\left[{a \atop b+1}\right](v|\tau) = e^{2i\pi a} \vartheta\left[{a \atop b}\right](v|\tau)\,, \tag{A.2}$$

$$\vartheta\left[{-a \atop -b}\right](v|\tau) = \vartheta\left[{a \atop b}\right](-v|\tau) \quad , \quad \vartheta\left[{a \atop b}\right](-v|\tau) = e^{4i\pi ab} \vartheta\left[{a \atop b}\right](v|\tau) \quad (a, b \in Z)\,. \tag{A.3}$$

We will use a modified Jacobi/Erderlyi notation where $\vartheta_1 = \vartheta\left[{1/2 \atop 1/2}\right]$, $\vartheta_2 = \vartheta\left[{1/2 \atop 0}\right]$, $\vartheta_3 = \vartheta\left[{0 \atop 0}\right]$, $\vartheta_4 = \vartheta\left[{0 \atop 1/2}\right]$.

Under modular transformations, the theta functions transform according to

$$\vartheta\left[{a \atop b}\right](v|\tau+1) = e^{-i\pi a(a-1)}\ \vartheta\left[{a \atop a+b-1/2}\right](v|\tau)\,, \tag{A.4}$$

$$\vartheta\left[{a \atop b}\right]\left(\frac{v}{\tau}| - \frac{1}{\tau}\right) = \sqrt{-i\tau}\ e^{2i\pi ab + i\pi \frac{v^2}{\tau}}\ \vartheta\left[{b \atop -a}\right](v|\tau)\,. \tag{A.5}$$

The Dedekind η-function of weight $\frac{1}{2}$ is related to the v-derivative of ϑ_1:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{A.6}$$

$$\frac{\partial}{\partial v}\vartheta_1(v)|_{v=0} \equiv \vartheta_1' = 2\pi\eta^3(\tau). \tag{A.7}$$

We can always set the variable v to zero by changing the shifts (a,b) appropriately:

$$\vartheta[^a_b]\,(v+\epsilon_1\tau+\epsilon_2|\tau) = e^{-i\pi\tau\epsilon_1^2-i\pi\epsilon_1(2v-b)-2i\pi\epsilon_1\epsilon_2}\ \vartheta[^{a-\epsilon_1}_{b-\epsilon_2}](v|\tau)\,. \tag{A.8}$$

In our conventions, we will systematically use shifts rather than the variable v.

We also note the following identities

$$\vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau) = 2\ \eta^3\,, \tag{A.9}$$

$$\vartheta_2^4(v|\tau) - \vartheta_1^4(v|\tau) = \vartheta_3^4(v|\tau) - \vartheta_4^4(v|\tau)\,, \tag{A.10}$$

We have the following identities for the derivatives of ϑ-functions

$$\partial_\tau(\frac{\vartheta_2}{\eta}) = \frac{i\pi}{12\eta}\left(\vartheta_3^4+\vartheta_4^4\right) \tag{A.11}$$

$$\partial_\tau(\frac{\vartheta_3}{\eta}) = \frac{i\pi}{12\eta}\left(\vartheta_2^4-\vartheta_4^4\right) \tag{A.12}$$

$$\partial_\tau(\frac{\vartheta_4}{\eta}) = \frac{i\pi}{12\eta}\left(-\vartheta_2^4-\vartheta_3^4\right) \tag{A.13}$$

Note that the above is valid for all rational values of a,b,h,g. The case $h,g\in\{0,1/2\}$can be seen as a special case, relevant for $\mathbb{Z}_2$-orbifolds, while $h,g\in\{0,1/n,\cdots(n-1)/n\}$ arise in the $\mathbb{Z}_n$-case (see, e.g., [27] or [28]).
We also use the short-hand notation

$$\vartheta[^a_b](\tau) := \vartheta[^a_b](0|\tau) \tag{A.14}$$

as well as

$$\vartheta[^a_b](m\cdot) := \vartheta[^a_b](0|m\tau) \tag{A.15}$$

Eisenstein series

The Eisenstein series E_{2n} are defined as

$$E_{2n} = 1 - \frac{4n}{B_{2n}}\sum_{k\geq 1}\frac{k^{2n-1}q^k}{1-q^k}. \tag{A.16}$$

E_{2n} with $n>1$ are holomorphic modular forms of weight $2n$. The Eisenstein series E_2 is often called quasi modular since under modular transformations, it transforms with a shift

$$E_2(-\frac{1}{\tau}) = \tau^2\left(E_2(\tau)+\frac{6}{\pi i\tau}\right). \tag{A.17}$$

Adding a term that compensates this shift yields the modular, but only "almost holomorphic" form of weight two $\widehat{E}_2$

$$\widehat{E}_2 = E_2 - \frac{3}{\pi\tau_2}. \tag{A.18}$$

The ring of almost holomorphic modular forms is generated by $\widehat{E}_2$ and the next two Eisenstein series

$$\begin{aligned} E_4 &= 1 + 240 \sum_{k\geq 1} \frac{k^3 q^k}{1-q^k} = \frac{1}{2} \sum_{a,b} \vartheta[{}^a_b]^8 \\ E_6 &= 1 - 504 \sum_{k\geq 1} \frac{k^5 q^k}{1-q^k}. \end{aligned} \tag{A.19}$$

Lie algebra lattice sums

Any shifted lattice sum over E_8 can be written in terms of theta functions as

$$\sum_{p\in\Gamma_{E_8}+a\gamma} q^{\frac{p^2}{2}} \mathrm{e}^{2\pi \mathrm{i} b p\cdot\gamma} = \sum_{\alpha,\beta} \prod_{i=1}^{8} \vartheta[{}^{\alpha+a\gamma_i}_{\beta+b\gamma_i}] \mathrm{e}^{-\pi \mathrm{i} \sum_i \gamma_i \beta a} \tag{A.20}$$

In particular,

$$E_4 = \frac{1}{2} \sum_{p\in\Gamma_{E_8}} q^{\frac{p^2}{2}} \tag{A.21}$$

and E_6 is related to the E_8 lattice shifted by any modular invariant embedding γ

$$E_6 = \sum_{(a,b)\neq(0,0)} \frac{c(a,b)}{2\vartheta[{}^{\frac{1}{2}+a}_{\frac{1}{2}+b}]\vartheta[{}^{\frac{1}{2}-a}_{\frac{1}{2}-b}]} \sum_{p\in\Gamma_{E_8}+a\gamma} q^{\frac{p^2}{2}} \mathrm{e}^{2\pi \mathrm{i} b p\cdot\gamma}, \tag{A.22}$$

with $c(a,b)$ as defined in section 3.
An obvious generalization of (A.20) is the modified Siegel-Narain Theta function over a general shifted lattice Γ of signature (b^+, b^-) with an insertion of $(p_R)^{2g-2}$

$$\Theta^g_\Gamma(\tau,\gamma,a,b) = \sum_{p\in\Gamma+a\gamma} (p_R)^{2g-2} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}} \mathrm{e}^{2\pi \mathrm{i} b\gamma\cdot p}. \tag{A.23}$$

We also use the notation

$$\Theta_\Gamma(\tau,\gamma_1,\gamma_2;P,\phi) = \sum_{p\in\Gamma+\gamma_1} \exp(-\frac{\Delta}{8\pi\tau_2})\phi(P(p)) q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}} \mathrm{e}^{2\pi \mathrm{i}\gamma_2\cdot p}, \tag{A.24}$$

where γ_1, γ_2 are shifts, P is an isometry from $\Gamma \times \mathbb{R}$ to $\mathbb{R}^{b^+,b^-}$, ϕ is a polynomial on $\mathbb{R}^{b^+,b^-}$ of degree m^+ in the first b^+ variables and of degree m^- in the others, and Δ is the Euclidean Laplacian on $\mathbb{R}^{b^+,b^-}$. The isometry P defines projections on $\mathbb{R}^+, \mathbb{R}^-$ written as $P_+(p) = p_R, \quad P_-(p) = p_L$. We will here only consider cases where the shifts are proportional, $\gamma_1 = a\gamma \sim \gamma_2 = b\gamma$.

B Lattice reduction

In [13], Borcherds developed the technique of lattice reduction to compute integrals of the form

$$\Phi_\Gamma = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F_M(\tau)\Theta_M(\tau,\gamma_1,\gamma_2;P,\phi), \tag{B.1}$$

where M is a lattice of signature (b^+,b^-), $\Theta_M(\tau,\gamma_1,\gamma_2;P,\phi)$ is the generalized Siegel theta function with projection P and polynomial insertion ϕ as defined in appendix A and F_M is a (quasi) modular form of weight $(-\frac{b^-}{2}-m^-,-\frac{b^+}{2}-m^+)$ that can be constructed from a (quasi) modular form F with weights $(\frac{b^+}{2}+m^+-\frac{b^-}{2}-m^-,0)$ as $F_M=\tau_2^{\frac{b^+}{2}+m^+}F$. The integral (B.1) can be decomposed into a sum over a reduced lattice K of signature (b^+-1,b^--1) and a new integral Φ_K involving K instead of M ([13], Theorem 7.1). Iterating this procedure, on arrives at an integral Φ_{K_f} with a lattice K_f of signature $(b^+-b^-,0)$ respectively $(0,b^--b^+)$ that can in principle be solved using standard methods.
The reduction steps proceed as follows. Choose two vectors z, z' in M with z primitive and $(z,z)=0$, $(z,z')=1$. The reduced lattice is then defined as $K=\frac{M\,\cap\,z^\perp}{\mathbb{Z}_z}$. We also define reduced projections $\tilde{P}$ in a natural way:

$$\tilde{P}_\pm(\lambda) = P_\pm(\lambda) - \frac{(P_\pm(\lambda),z_\pm)}{z_\pm^2}z_\pm. \tag{B.2}$$

We can then expand the polynomial ϕ in terms of $(\lambda,z_\pm)$ as

$$\phi(P(\lambda)) = \sum_{h^+,h^-} = (\lambda,z_+)^{h^+}(\lambda,z_-)^{h^-}\phi_{h^+,h^-}(\tilde{P}(\lambda)). \tag{B.3}$$

The statement of Borcherds' theorem is then that with these conventions, z_+^2 *sufficiently small* and $\tilde{P}_+(\lambda_K)\neq 0$, Φ_M is given by

$$\begin{aligned}&\frac{\sqrt{2}}{|z_+|}\sum_{h\geq 0}\sum_{h^+,h^-}\frac{h!(-z_+^2/\pi)^h}{(2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}\sum_j\sum_{\lambda_K\in K}\frac{(-\Delta)^j(\bar{\phi}_{h^+,h^-})(\tilde{P}(\lambda))}{(8\pi)^j j!}\\ &\cdot\sum_{l,t} q^{l(\lambda_K,(-z'+\frac{z_+}{2z_+^2}+\frac{z_-}{2z_-^2}))}c(\lambda_K^2,t)l^{h^++h^--2h}\left(\frac{l}{2|z_+||\tilde{P}_+(\lambda_K)|}\right)^{h-h^+-h^--j-t+\frac{b^+}{2}+m^+-3/2}\\ &K_{h-h^+-h^--j-t-b^+/2+m^+-3/2}\left(\frac{2\pi l|\tilde{P}_+(\lambda_K)|}{|z_+|}\right).\end{aligned} \tag{B.4}$$

For $\tilde{P}_+(\lambda)=0$, the last two factors have to be replaced by the analytic continuation at $\epsilon=0$ of

$$\begin{aligned}&\left(\frac{\pi l^2}{2z_+^2}\right)^{h-h^+-h^--j-t+b^+/2+m^+-3/2-\epsilon}\\ &\cdot\Gamma(-h+h^++h^-+j+t-b^+/2-m^++3/2+\epsilon).\end{aligned} \tag{B.5}$$

C Instanton tables and heterotic-type II duals

Table C.1 lists the dual K3-fibrations for the $\mathbb{Z}_N$-orbifolds defined in table 2.1 [8].

Type	Group	(n_h, n_v)	CY weights
$\mathbb{Z}_2, 3+8$ WL	$SU(4) \times E_8' \times U(1)^4$	$(167, 15)$	$(1, 1, 12, 16, 18, 20)$
$\mathbb{Z}_2, 2+8$ WL	$SU(3) \times E_8' \times U(1)^4$	$(230, 14)$	$(1, 1, 12, 16, 18)$
$\mathbb{Z}_2, 1+8$ WL	$SU(2) \times E_8' \times U(1)^4$	$(319, 13)$	$(1, 1, 12, 16, 30)$
$\mathbb{Z}_2, 0+8$ WL	$E_8' \times U(1)^4$	$(492, 12)$	$(1, 1, 12, 28, 42)$
$\mathbb{Z}_3, 3+6$ WL	$SU(4) \times E_6' \times U(1)^4$	$(129, 13)$	$(1, 1, 6, 10, 12, 14)$
$\mathbb{Z}_3, 2+6$ WL	$SU(3) \times E_6' \times U(1)^4$	$(168, 12)$	$(1, 1, 6, 10, 12)$
$\mathbb{Z}_3, 1+6$ WL	$SU(2) \times E_6' \times U(1)^4$	$(221, 11)$	$(1, 1, 6, 10, 18)$
$\mathbb{Z}_3, 0+6$ WL	$E_6' \times U(1)^4$	$(322, 10)$	$(1, 1, 6, 16, 24)$
$\mathbb{Z}_4, 3+4$ WL	$SU(4) \times SO(8)' \times U(1)^4$	$(123, 11)$	$(1, 1, 4, 8, 10, 12)$
$\mathbb{Z}_4, 2+4$ WL	$SU(3) \times SO(8)' \times U(1)^4$	$(154, 10)$	$(1, 1, 4, 8, 10)$
$\mathbb{Z}_4, 1+4$ WL	$SU(2) \times SO(8)' \times U(1)^4$	$(195, 9)$	$(1, 1, 4, 8, 14)$
$\mathbb{Z}_4, 0+4$ WL	$SO(8)' \times U(1)^4$	$(272, 8)$	$(1, 1, 4, 12, 18)$
$\mathbb{Z}_6, 3+0$ WL	$SU(4) \times E_6' \times U(1)^4$	$(139, 7)$	$(1, 1, 2, 6, 8, 10)$
$\mathbb{Z}_6, 2+0$ WL	$SU(3) \times E_6' \times U(1)^4$	$(162, 6)$	$(1, 1, 2, 6, 8)$
$\mathbb{Z}_6, 1+0$ WL	$SU(2) \times E_6' \times U(1)^4$	$(191, 5)$	$(1, 1, 2, 6, 10)$
$\mathbb{Z}_6, 0+0$ WL	$E_6' \times U(1)^4$	$(244, 4)$	$(1, 1, 2, 8, 12)$

Table C.1: The chains of heterotic-type II duals studied in [8]

Tables C.2–C.15 give instanton numbers at $g = 0, \cdots 4$ for the $\mathbb{Z}_{2,3,4,6}$ orbifolds defined in table 2.1.

g	$\frac{p^2}{2}=-1$	0	1	2	3	4	5	6
0	-2	960	56808	1364480	20920140	240357888	2244734960	17884219392
1	0	4	-1908	-119360	-3077460	-50495040	-617959240	-6118785792
2	0	0	-6	2848	185694	5045376	87240260	1122823296
3	0	0	0	8	-3780	-255792	-7276660	-131766240
4	0	0	0	0	-10	4704	329630	9782592

Table C.2: $\mathbb{Z}_2$, 8 Wilson lines, dual to $X^{1,1,12,28,42}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{4}$	0	$\frac{3}{4}$	1	$\frac{7}{4}$	2	$\frac{11}{4}$	3	$\frac{15}{4}$
0	-2	176	612	12672	30240	320976	661696	5031040	9509328	58372272
1	0	0	4	-352	-1212	-26400	-64136	-719392	-1509700	-12091776
2	0	0	0	0	-6	528	1804	40832	100422	1173600
3	0	0	0	0	0	0	8	-704	-2388	-55968
4	0	0	0	0	0	0	0	0	-10	880

Table C.3: $\mathbb{Z}_2$, 8+1 Wilson lines, dual to $X^{1,1,12,16,30}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{3}$	0	$\frac{2}{3}$	1	$\frac{5}{3}$	2	$\frac{8}{3}$	3	$\frac{11}{3}$
0	-2	90	432	5904	18252	142146	365600	2144016	4936140	24107760
1	0	0	4	-180	-852	-12348	-39080	-320436	-844140	-5189400
2	0	0	0	0	-6	270	1264	19152	61578	524952
3	0	0	0	0	0	0	8	-360	-1668	-26316
4	0	0	0	0	0	0	0	0	-10	450

Table C.4: $\mathbb{Z}_2$, 8+2 Wilson lines, dual to $X^{1,1,12,16,30}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{2}$	$-\frac{3}{8}$	0	$\frac{1}{2}$	$\frac{5}{8}$	1	$\frac{3}{2}$	$\frac{13}{8}$	2	$\frac{5}{2}$
0	-2	28	64	304	2144	3392	11412	52144	75136	211040	781312
1	0	0	0	4	-56	-128	-596	-4456	-7168	-24632	-117376
2	0	0	0	0	0	0	-6	84	192	880	6880
3	0	0	0	0	0	0	0	0	0	8	-112

Table C.5: $\mathbb{Z}_2$, 8+3 Wilson lines, dual to $X^{1,1,12,16,18}$

g	$\frac{p^2}{2}=-1$	$-\frac{3}{5}$	$-\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{3}{5}$	1	$\frac{7}{5}$	$\frac{8}{5}$	2	$\frac{12}{5}$
0	-2	14	52	200	1020	2158	7068	23916	43080	122840	347376
1	0	0	0	4	-28	-104	-388	-2124	-4628	-15320	-54064
2	0	0	0	0	0	0	-6	42	156	568	3284
3	0	0	0	0	0	0	0	0	0	8	-56

Table C.6: $\mathbb{Z}_2$, 8+4 Wilson lines, dual to $X^{1,1,12,16,18,20}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{2}$	$-\frac{3}{8}$	0	$\frac{1}{2}$	$\frac{5}{8}$	1	$\frac{3}{2}$	$\frac{13}{8}$	2
0	-2	8	24	264	9104	17272	86292	634464	1009936	3647120
1	0	0	0	4	-16	-48	-516	-18256	-34688	-174152
2	0	0	0	0	0	0	-6	72	760	27440
3	0	0	0	0	0	0	0	0	0	8

Table C.7: $\mathbb{Z}_6$, 3 Wilson lines, dual to $X^{1,1,2,6,8,10}$

g	$\frac{p^2}{2}=-1$	0	1	2	3	4	5	6
0	-2	624	54792	1609088	28265184	360251424	3659578208	31296575232
1	0	4	-1236	-113312	-3551892	-66631944	-903741184	-9729986112
2	0	0	-6	1840	174270	5731824	113066144	1610777952
3	0	0	0	8	-2436	-237648	-8154292	-168125136
4	0	0	0	0	-10	3024	303422	10826544

Table C.8: $\mathbb{Z}_3$, 6 Wilson lines, dual to $X^{1,1,6,16,24}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{4}$	0	$\frac{3}{4}$	1	$\frac{7}{4}$	2	$\frac{11}{4}$	3
0	-2	104	420	11856	30240	373464	801472	6750016	13138500
1	0	0	4	-208	-828	-24336	-62984	-818896	-1787716
2	0	0	0	0	-6	312	1228	37232	97350
3	0	0	0	0	0	0	8	-416	-1620

Table C.9: $\mathbb{Z}_3$, 6+1 Wilson lines, dual to $X^{1,1,6,10,18}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{3}$	0	$\frac{2}{3}$	1	$\frac{5}{3}$	2	$\frac{8}{3}$	3
0	-2	54	312	5616	18900	167778	454688	2914704	6972912
1	0	0	4	-108	-612	-11556	-39656	-369684	-1025244
2	0	0	0	0	-6	162	904	17712	61602
3	0	0	0	0	0	0	8	-216	-1188

Table C.10: $\mathbb{Z}_3$, 6+2 Wilson lines, dual to $X^{1,1,6,10,12}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{2}$	$-\frac{3}{8}$	0	$\frac{1}{2}$	$\frac{5}{8}$	1	$\frac{3}{2}$	$\frac{13}{8}$	2	$\frac{5}{2}$
0	-2	16	40	232	2024	3320	12228	61600	90592	269456	1065784
1	0	0	0	4	-32	-80	-452	-4144	-6880	-25832	-135472
2	0	0	0	0	0	0	-6	48	120	664	6328
3	0	0	0	0	0	0	0	0	0	8	-64

Table C.11: $\mathbb{Z}_3$, 6+3 Wilson lines, dual to $X^{1,1,6,10,12}$

g	$\frac{p^2}{2}=-1$	0	1	2	3	4	5
0	-2	528	90036	3679520	80559180	1212246784	14073864648
1	0	4	-1044	-183224	-7903452	-183923136	-2938551600
2	0	0	-6	1552	278466	12502704	304651808
3	0	0	0	8	-2052	-375744	-17481820
4	0	0	0	0	-10	2544	475034

Table C.12: $\mathbb{Z}_4$, 4 Wilson lines, dual to $X^{1,1,4,12,18}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{4}$	0	$\frac{3}{4}$	1	$\frac{7}{4}$	2	$\frac{11}{4}$	3
0	-2	80	372	18432	52428	832848	1908808	18982912	38738880
1	0	0	4	-160	-732	-37344	-107072	-1776928	-4135132
2	0	0	0	0	-6	240	1084	56576	163146
3	0	0	0	0	0	0	8	-320	-1428

Table C.13: $\mathbb{Z}_4$, 4+1 Wilson lines, dual to $X^{1,1,4,8,14}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{3}$	0	$\frac{2}{3}$	1	$\frac{5}{3}$	2	$\frac{8}{3}$	3
0	-2	42	288	8928	34488	381894	1127168	8355360	21263796
1	0	0	4	-84	-564	-18108	-70688	-817692	-2463540
2	0	0	0	0	-6	126	832	27456	107982
3	0	0	0	0	0	0	8	-168	-1092

Table C.14: $\mathbb{Z}_4$, 4+2 Wilson lines, dual to $X^{1,1,4,8,10}$

g	$\frac{p^2}{2}=-1$	$-\frac{1}{2}$	$-\frac{3}{8}$	0	$\frac{1}{2}$	$\frac{5}{8}$	1	$\frac{3}{2}$	$\frac{13}{8}$	2	$\frac{5}{2}$
0	-2	12	32	224	3136	5536	23392	139688	213248	694400	3063424
1	0	0	0	4	-24	-64	-436	-6344	-11264	-48112	-298288
2	0	0	0	0	0	0	-6	36	96	640	9600
3	0	0	0	0	0	0	0	0	0	8	-48

Table C.15: $\mathbb{Z}_4$, 4+3 Wilson lines, dual to $X^{1,1,4,8,10,12}$

References

[1] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "Topological amplitudes in string theory," *Nucl. Phys.* **B413** (1994) 162–184, hep-th/9307158.

[2] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, hep-th/9309140.

[3] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "N=2 type II heterotic duality and higher derivative F terms," *Nucl. Phys.* **B455** (1995) 109–130, hep-th/9507115.

[4] J. A. Harvey and G. W. Moore, "Algebras, BPS States, and Strings," *Nucl. Phys.* **B463** (1996) 315–368, hep-th/9510182.

[5] M. Mariño and G. W. Moore, "Counting higher genus curves in a Calabi-Yau manifold," *Nucl. Phys.* **B543** (1999) 592–614, hep-th/9808131.

[6] A. Klemm, M. Kreuzer, E. Riegler, and E. Scheidegger, "Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections," *JHEP* **05** (2005) 023, hep-th/0410018.

[7] A. Klemm and M. Mariño, "Counting BPS states on the Enriques Calabi-Yau," hep-th/0512227.

[8] G. Aldazabal, A. Font, L. E. Ibáñez, and F. Quevedo, "Chains of N=2, D=4 heterotic/type II duals," *Nucl. Phys.* **B461** (1996) 85–100, hep-th/9510093.

[9] A. Ceresole, R. D'Auria, S. Ferrara, and A. Van Proeyen, "Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity," *Nucl. Phys.* **B444** (1995) 92–124, hep-th/9502072.

[10] B. de Wit, V. Kaplunovsky, J. Louis, and D. Lust, "Perturbative couplings of vector multiplets in N=2 heterotic string vacua," *Nucl. Phys.* **B451** (1995) 53–95, hep-th/9504006.

[11] P. Mayr and S. Stieberger, "Moduli dependence of one loop gauge couplings in (0,2) compactifications," *Phys. Lett.* **B355** (1995) 107–116, hep-th/9504129.

[12] G. Lopes Cardoso, G. Curio, and D. Lust, "Perturbative couplings and modular forms in N = 2 string models with a Wilson line," *Nucl. Phys.* **B491** (1997) 147–183, hep-th/9608154.

[13] R. E. Borcherds, "Automorphic forms with singularities on Grassmannians," *Invent. Math.* **132** (1998) 491–562, alg-geom/9609022.

[14] P. Berglund, S. H. Katz, A. Klemm, and P. Mayr, "New Higgs transitions between dual N = 2 string models," *Nucl. Phys.* **B483** (1997) 209–228, hep-th/9605154.

[15] T. Kawai, "$N = 2$ heterotic string threshold correction, $K3$ surface and generalized Kac-Moody superalgebra," *Phys. Lett.* **B372** (1996) 59–64, hep-th/9512046.

[16] S. Stieberger, "(0,2) heterotic gauge couplings and their M-theory origin," *Nucl. Phys.* **B541** (1999) 109–144, hep-th/9807124.

[17] M. Henningson and G. W. Moore, "Threshold corrections in K(3) x T(2) heterotic string compactifications," *Nucl. Phys.* **B482** (1996) 187–212, hep-th/9608145.

[18] S. Kachru and C. Vafa, "Exact results for N = 2 compactifications of heterotic strings," *Nucl. Phys. Proc. Suppl.* **46** (1996) 210–224.

[19] J. Polchinski, *String Theory*, vol. I&II. Cambridge University Press, 1998.

[20] B. R. Greene, D. R. Morrison, and A. Strominger, "Black hole condensation and the unification of string vacua," *Nucl. Phys.* **B451** (1995) 109–120, hep-th/9504145.

[21] J. F. Morales and M. Serone, "Higher derivative F-terms in N = 2 strings," *Nucl. Phys.* **B481** (1996) 389–402, hep-th/9607193.

[22] W. Lerche, B. E. W. Nilsson, A. N. Schellekens, and N. P. Warner, "Anomaly cancelling terms from the elliptic genus," *Nucl. Phys.* **B299** (1988) 91.

[23] E. Kiritsis, "Introduction to superstring theory," hep-th/9709062.

[24] W. Lerche, "Elliptic index and superstring effective actions," *Nucl. Phys.* **B308** (1988) 102.

[25] T. W. Grimm, A. Klemm, M. Mariño, and M. Weiss, "Direct integration of the topological string," hep-th/0702187.

[26] R. Gopakumar and C. Vafa, "M-theory and topological strings. I&II," hep-th/9809187, hep-th/9812127.

[27] D. Mumford, *Tata lectures on Theta*, vol. I,II, and III. Birkhaeuser, 1983.

[28] L. Alvarez-Gaumé, G. W. Moore, and C. Vafa, "Theta functions, modular invariance, and strings," *Commun. Math. Phys.* **106** (1986) 1–40.

[29] D. Maulik and R. Pandharipande, "A topological view of Gromov-Witten theory," arXiv:math.ag/0412503.

[30] D. Maulik and R. Pandharipande, "Gromov-Witten theory and Noether-Lefschetz theory," arXiv:0705.1653v1.